

Consistent testing for a constant copula under strong mixing based on the tapered block multiplier technique

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Abstract

Considering multivariate strongly mixing time series, nonparametric tests for a constant copula with specified or unspecified change point (candidate) are derived; the tests are consistent against general alternatives. A tapered block multiplier technique based on serially dependent multiplier random variables is provided to estimate p-values of the test statistics. Size and power of the tests in finite samples are evaluated with Monte Carlo simulations. The block multiplier technique might have several other applications for statistical inference on copulas of serially dependent data.

Key words: Change point test; Copula; Empirical copula process; Nonparametric estimation; Time series; Strong mixing; Multiplier central limit theorem.

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1. Introduction

Over the last decade, copulas have become a standard tool in modern risk management. The copula of a continuous random vector is a function which uniquely determines the dependence structure linking the marginal distribution functions. Copulas play a pivotal role for, e.g., measuring multivariate association [see 46], pricing multivariate options [see 49] and allocating financial assets [see 33]. The latter two references emphasize that time variation of copulas possesses an important impact on financial engineering applications.

Evidence for time-varying dependence structures can indirectly be drawn from functionals of the copula, e.g., Spearman's ρ , as suggested by Gaïffer et al. [20] and Wied et al. [52]. Investigating time variation of the copula itself, Busetti and Harvey [9] consider a nonparametric quantile-based test for a pointwise constant copula. Semiparametric tests for time variation of the parameter within a prespecified parametric copula model are proposed by Dias and Embrechts [15] and Giacomini et al. [23]. Guegan and Zhang [24] combine tests for constancy of the copula (on a given set of vectors on its domain), the copula family, and the parameter. All the latter references on time variation of the copula are based on the assumption of i.i.d. observations. With respect to financial time-series, this assumption may be approximated by the estimation of a GARCH model and by using the residuals obtained after GARCH filtration. However, the effect of replacing unobserved innovations by estimated residuals has to be taken into account. Therefore, specific techniques for residuals are required (cf. [12]) and exploring this approach, Rémillard [38] investigates a nonparametric change point test for the copula of residuals in stochastic volatility models.

In the present paper we go beyond these approaches and work in a purely nonparametric setting and under the allowance of general serial dependence of a multivariate time series measured by their alpha-mixing coefficients. In similar settings, Fermanian and Scaillet [19], Doukhan et al. [17] and Bücher and Volgushev [6] consider the nonparametric estimation of copulas. These references form the basis for our new tests for a constant copula under strong mixing assumptions. We introduce nonparametric Cramér-von Mises-, Kuiper-, and Kolmogorov-Smirnov tests which assess constancy of the copula on its entire domain. In consequence, they are consistent under general alternatives. Depending on the object of investigation, tests with a specified or unspecified change point (candidate) are introduced. Whereas the former setting requires a hypothesis on the change point location, it allows us to relax the assumption of strictly stationary univariate processes. P-values of the tests are estimated based on a generalization of the multiplier bootstrap technique introduced in Rémillard and Scaillet [40] to the case of strongly mixing time series. This new technique can be used to generalize several statistical inference methods from the i.i.d. case to the serial dependent case. The idea of our approach is comparable to block bootstrap methods with the difference that, instead of sampling blocks with replacement, we generate blocks of serially dependent multiplier random variables. For a general introduction to the latter idea, we refer to Bühlmann [7] and Paparoditis and Politis [31]. Independently from our work, a current working paper of van Kampen and Wied [51] investigates similar Kolmogorov-Smirnov tests for constancy of the copula with unspecified change point based on a bootstrap technique introduced in Inoue [26].

This paper is organized as follows. In Section 2, we discuss weak convergence of the empirical copula process under strong mixing. We derive a tapered block multiplier bootstrap technique for inference on the weak limit and assess it in finite samples. Tests for a constant copula with specified or unspecified change point (candidate) which are relying on this technique are established in Section 3. Section 4 concludes the paper. All the proofs

are deferred to Appendix A.

2. Nonparametric inference based on serially dependent observations

As a basis for the tests introduced in Section 3, we briefly recapitulate the results of Doukhan et al. [17], Segers [47] and Bücher and Volgushev [6] on the asymptotic behavior of the empirical copula process under nonrestrictive smoothness assumptions in the case of strongly mixing observations. The main result of this section is a multiplier-based resampling method for this particular setting. We establish its asymptotic behavior and investigate the performance in finite samples.

2.1. Asymptotic theory

Consider a vector-valued *process* $(\mathbf{X}_j)_{j \in \mathbb{Z}}$ with $\mathbf{X}_j = (X_{j,1}, \dots, X_{j,d})$ taking values in \mathbb{R}^d . Let F_i be the distribution function of $X_{j,i}$ for all $j \in \mathbb{Z}$, $i = 1, \dots, d$ and let F be the joint distribution of \mathbf{X}_j for all $j \in \mathbb{Z}$. Assume that all marginal distribution functions are continuous. Then, according to Sklar's Theorem [48], there exists a unique copula C such that $F(x_1, \dots, x_d) = C(F_1(x_1), \dots, F_d(x_d))$ for all $(x_1, \dots, x_d) \in \mathbb{R}^d$. The σ -fields generated by $\mathbf{X}_j, j \leq t$, and $\mathbf{X}_j, j \geq t$, are denoted by $\mathcal{F}_t = \sigma\{\mathbf{X}_j, j \leq t\}$ and $\mathcal{F}^t = \sigma\{\mathbf{X}_j, j \geq t\}$, respectively. We define

$$\alpha(\mathcal{F}_s, \mathcal{F}^{s+r}) = \sup_{A \in \mathcal{F}_s, B \in \mathcal{F}^{s+r}} |P(A \cap B) - P(A)P(B)|.$$

The strong- (or α -) mixing coefficient $\alpha_{\mathbf{X}}$ corresponding to the process $(\mathbf{X}_j)_{j \in \mathbb{Z}}$ is given by $\alpha_{\mathbf{X}}(r) = \sup_{s \geq 0} \alpha(\mathcal{F}_s, \mathcal{F}^{s+r})$. The process $(\mathbf{X}_j)_{j \in \mathbb{Z}}$ is said to be *strongly mixing* if $\alpha_{\mathbf{X}}(r) \rightarrow 0$ for $r \rightarrow \infty$. This type of weak dependence covers a broad range of time-series models. Consider the following examples, cf. Doukhan [16] and Carrasco and Chen [11]:

Example 1. *i) AR(1) processes $(X_j)_{j \in \mathbb{Z}}$ given by*

$$X_j = \beta X_{j-1} + \varepsilon_j,$$

where $(\varepsilon_j)_{j \in \mathbb{Z}}$ is a sequence of independent and identically distributed continuous innovations with mean zero. For $|\beta| < 1$, the process is strictly stationary and strongly mixing with exponential decay of $\alpha_{\mathbf{X}}(r)$.

ii) GARCH(1,1) processes $(X_j)_{j \in \mathbb{Z}}$,

$$X_j = \sigma_j \varepsilon_j, \quad \sigma_j^2 = \omega + \beta \sigma_{j-1}^2 + \alpha \varepsilon_{j-1}^2, \quad (1)$$

where $(\varepsilon_j)_{j \in \mathbb{Z}}$ is a sequence of independent and identically distributed continuous innovations, independent of σ_0^2 , with mean zero and variance one. For $\alpha + \beta < 1$, the process is strictly stationary and strongly mixing with exponential decay of $\alpha_{\mathbf{X}}(r)$.

iii) For multivariate analogues of i) and ii) see Section 2.3 below.

Let $\mathbf{X}_1, \dots, \mathbf{X}_n$ denote a sample from $(\mathbf{X}_j)_{j \in \mathbb{Z}}$. A simple nonparametric estimator for the copula C is given by the *empirical copula* which is first considered by Rüschendorf [44] and Deheuvels [14]. Depending on whether the marginal distribution functions are assumed to be known or unknown, we define, for $\mathbf{u} \in [0, 1]^d$,

$$C_n(\mathbf{u}) := \frac{1}{n} \sum_{j=1}^n \mathbf{1}_{\{\mathbf{U}_j \leq \mathbf{u}\}}, \quad \hat{C}_n(\mathbf{u}) := \frac{1}{n} \sum_{j=1}^n \mathbf{1}_{\{\hat{\mathbf{U}}_j \leq \mathbf{u}\}}, \quad (2)$$

where $\mathbf{U}_j = (U_{j,1}, \dots, U_{j,d})$ and $\widehat{\mathbf{U}}_j = (\widehat{U}_{j,1}, \dots, \widehat{U}_{j,d})$ with observations $U_{j,i} = F_i(X_{j,i})$ and *pseudo-observations* $\widehat{U}_{j,i} = \widehat{F}_i(X_{j,i})$ for all $j = 1, \dots, n$ and $i = 1, \dots, d$, and where $\widehat{F}_i(x) = n^{-1} \sum_{j=1}^n \mathbf{1}_{\{X_{j,i} \leq x\}}$ for all $x \in \mathbb{R}$. Unless otherwise noted, the marginal distribution functions are assumed to be unknown and the *empirical copula* \widehat{C}_n is used. In addition to the practical relevance of this assumption, Genest and Segers [22] prove that pseudo-observations $\widehat{\mathbf{U}}_j$ permit more efficient inference on the copula than observations \mathbf{U}_j for a broad class of copulas.

Doukhan et al. [17] investigate dependent observations and establish the asymptotic behavior of the *empirical copula process*, defined by $\sqrt{n}\{\widehat{C}_n - C\}$, assuming the copula to possess continuous partial derivatives on $[0, 1]^d$. Segers [47] points out that many popular families of copulas (e.g., the Gaussian, Clayton, and Gumbel–Hougaard families) do not satisfy the assumption of continuous first partial derivatives on $[0, 1]^d$. He establishes the asymptotic behavior of the empirical copula for serially independent observations under the weaker condition

$$D_i C(\mathbf{u}) \text{ exists and is continuous on } \left\{ \mathbf{u} \in [0, 1]^d \mid u_i \in (0, 1) \right\} \text{ for all } i = 1, \dots, d. \quad (3)$$

Under Condition (3), the partial derivatives' domain can be extended to $\mathbf{u} \in [0, 1]^d$ by

$$D_i C(\mathbf{u}) = \begin{cases} \lim_{h \rightarrow 0} \frac{C(\mathbf{u} + h\mathbf{e}_i) - C(\mathbf{u})}{h} & \text{for all } u \in [0, 1]^d, \ 0 < u_i < 1, \\ \limsup_{h \downarrow 0} \frac{C(\mathbf{u} + h\mathbf{e}_i) - C(\mathbf{u})}{h} & \text{for all } u \in [0, 1]^d, \ u_i = 0, \\ \limsup_{h \downarrow 0} \frac{C(\mathbf{u}) - C(\mathbf{u} - h\mathbf{e}_i)}{h} & \text{for all } u \in [0, 1]^d, \ u_i = 1, \end{cases} \quad (4)$$

and for all $i = 1, \dots, d$, where \mathbf{e}_i denotes the i th column of a $d \times d$ identity matrix. The following generalization of the results in [17] and [47] is a consequence of Theorem 2.4 in Bücher and Volgushev [6], see Corollary 2.5 in that reference.

Theorem 1. *Consider observations $\mathbf{X}_1, \dots, \mathbf{X}_n$, drawn from a strictly stationary process $(\mathbf{X}_j)_{j \in \mathbb{Z}}$ satisfying the strong mixing condition $\alpha_{\mathbf{X}}(r) = \mathcal{O}(r^{-a})$ for some $a > 1$. Then*

$$\mathbb{B}_{C,n} = \sqrt{n}(C_n - C) \xrightarrow{w} \mathbb{B}_C$$

in the metric space space of uniformly bounded functions on $[0, 1]^d$ equipped with the uniform metric $(\ell^\infty([0, 1]^d), \|\cdot\|_\infty)$. Here, \mathbb{B}_C denotes a centered tight Gaussian process on $[0, 1]^d$ with covariance function

$$\gamma(\mathbf{u}, \mathbf{v}) = \text{Cov}(\mathbb{B}_C(\mathbf{u}), \mathbb{B}_C(\mathbf{v})) = \sum_{j \in \mathbb{Z}} \text{Cov}(\mathbf{1}_{\{\mathbf{U}_0 \leq \mathbf{u}\}}, \mathbf{1}_{\{\mathbf{U}_j \leq \mathbf{v}\}}) \text{ for all } \mathbf{u}, \mathbf{v} \in [0, 1]^d. \quad (5)$$

Moreover, if C satisfies Condition (3), then

$$\widehat{\mathbb{G}}_{C,n} = \sqrt{n}(\widehat{C}_n - C) \xrightarrow{w} \mathbb{G}_C$$

in $\ell^\infty([0, 1]^d)$, where \mathbb{G}_C represents a Gaussian process given by

$$\mathbb{G}_C(\mathbf{u}) = \mathbb{B}_C(\mathbf{u}) - \sum_{i=1}^d D_i C(\mathbf{u}) \mathbb{B}_C(\mathbf{u}^{(i)}) \text{ for all } \mathbf{u} \in [0, 1]^d. \quad (6)$$

Here, $\mathbf{u}^{(i)}$ denotes the vector where all coordinates, except the i th coordinate of \mathbf{u} , are replaced by 1.

Notice that the covariance structure as given in Equation (5) depends on the entire process $(\mathbf{X}_j)_{j \in \mathbb{Z}}$ in case it is not serially uncorrelated.

2.2. Resampling techniques

In this Section, we introduce two bootstrap techniques for the empirical copula process which are applicable in the case of strongly mixing observations. We begin with the (moving) block bootstrap, which serves as a benchmark in the finite sample assessment. Subsequently, we derive the main result of this Section about the asymptotic consistency of a generalized multiplier bootstrap technique.

2.2.1. The block bootstrap

Fermanian et al. [18] investigate the empirical copula process for independent and identically distributed observations $\mathbf{X}_1, \dots, \mathbf{X}_n$ and prove consistency of the *nonparametric bootstrap* method which is based on sampling with replacement from $\mathbf{X}_1, \dots, \mathbf{X}_n$. We denote a bootstrap sample by $\mathbf{X}_1^B, \dots, \mathbf{X}_n^B$ and define

$$\hat{C}_n^B(\mathbf{u}) := \frac{1}{n} \sum_{j=1}^n \mathbf{1}_{\{\hat{\mathbf{U}}_j^B \leq \mathbf{u}\}} \text{ for all } \mathbf{u} \in [0, 1]^d, \quad \hat{U}_{j,i}^B := \frac{1}{n} \sum_{k=1}^n \mathbf{1}_{\{X_{k,i}^B \leq X_{j,i}^B\}}$$

for all $j = 1, \dots, n$ and $i = 1, \dots, d$. Notice that the bootstrap empirical copula can equivalently be expressed based on multinomially $(n, n^{-1}, \dots, n^{-1})$ distributed random variables $\mathbf{W} = (W_1, \dots, W_n)$:

$$\hat{C}_n^W(\mathbf{u}) := \frac{1}{n} \sum_{j=1}^n W_j \mathbf{1}_{\{\hat{\mathbf{U}}_j^W \leq \mathbf{u}\}} \text{ for all } \mathbf{u} \in [0, 1]^d, \quad \hat{U}_{j,i}^W := \frac{1}{n} \sum_{k=1}^n W_k \mathbf{1}_{\{X_{k,i} \leq X_{j,i}\}}$$

for all $j = 1, \dots, n$ and $i = 1, \dots, d$. It is shown in [18, 3] that the corresponding *bootstrap empirical copula process converges weakly conditional on $\mathbf{X}_1, \dots, \mathbf{X}_n$ in probability* in $(\ell^\infty([0, 1]^d), \|\cdot\|_\infty)$, notationally

$$\hat{\mathbb{G}}_{C,n}^W = \sqrt{n}(\hat{C}_n^W - C_n) \xrightarrow[W]{\mathbb{P}} \mathbb{G}_C. \quad (7)$$

Here, *weak convergence conditional on the data in probability* is understood in the Hoffmann-Jørgensen sense as defined in Kosorok [29], i.e., $\hat{\mathbb{G}}_{C,n}^W \xrightarrow[W]{\mathbb{P}} \mathbb{G}_C$ if and only if the following two conditions hold:

$$\sup_{h \in BL_1(\ell^\infty([0, 1]^d))} \left| \mathbb{E}_W h(\hat{\mathbb{G}}_{C,n}^W) - \mathbb{E} h(\mathbb{G}_C) \right| \xrightarrow{\mathbb{P}} 0, \quad (8)$$

$$\mathbb{E}_W h(\hat{\mathbb{G}}_{C,n}^W)^* - \mathbb{E}_W h(\hat{\mathbb{G}}_{C,n}^W)_* \xrightarrow{\mathbb{P}} 0 \text{ for all } h \in BL_1(\ell^\infty([0, 1]^d)). \quad (9)$$

Here, $\xrightarrow{\mathbb{P}}$ denotes convergence in outer probability and \mathbb{E}_W denotes expectation with respect to \mathbf{W} conditional on $\mathbf{X}_1, \dots, \mathbf{X}_n$. Furthermore, $BL_1(\ell^\infty([0, 1]^d))$ denotes the set

$$\left\{ f : \ell^\infty([0, 1]^d) \rightarrow \mathbb{R} \mid \|f\|_\infty \leq 1, |f(\beta) - f(\gamma)| \leq \|\beta - \gamma\|_\infty \text{ for all } \gamma, \beta \in \ell^\infty([0, 1]^d) \right\}.$$

of all Lipschitz-continuous functions bounded by 1 with Lipschitz-constant not exceeding 1 and the asterisks in (9) denote measurable majorants and minorants with respect to the joint data (i.e., $\mathbf{X}_1, \dots, \mathbf{X}_n$ and \mathbf{W}). Weak convergence conditional on $\mathbf{X}_1, \dots, \mathbf{X}_n$ almost surely is defined analogously by replacing outer probability convergence in (8) and (9) by

outer almost sure convergence. Due to the lack of a general continuous mapping Theorem and an easy functional delta method, see [50, 29], we do not consider the outer almost sure version in this paper.

Whereas the bootstrap is consistent for i.i.d. samples, consistency generally fails for serially dependent samples. Therefore, a *block bootstrap method* is proposed by Künsch [30]. Given the sample $\mathbf{X}_1, \dots, \mathbf{X}_n$, the block bootstrap method requires blocks of size $l_B = l_B(n)$, $l_B(n) \rightarrow \infty$ as $n \rightarrow \infty$ and $l_B(n) = o(n)$, consisting of consecutive observations

$$B_{h,l_B} = \{\mathbf{X}_{h+1}, \dots, \mathbf{X}_{h+l_B}\}, \text{ for all } h = 0, \dots, n - l_B.$$

We assume $n = kl_B$ (otherwise the last block is truncated) and simulate $\mathbf{H} = (H_1, \dots, H_k)$ independent and uniformly distributed random variables on $\{0, \dots, n - l_B\}$. The block bootstrap sample is given by the observations of the k blocks $B_{H_1, l_B}, \dots, B_{H_k, l_B}$, i.e.,

$$\mathbf{X}_{H_1+1}, \dots, \mathbf{X}_{H_1+l_B}, \mathbf{X}_{H_2+1}, \dots, \mathbf{X}_{H_2+l_B}, \dots, \mathbf{X}_{H_k+1}, \dots, \mathbf{X}_{H_k+l_B}.$$

Denote the block bootstrap empirical copula based on this sample by $\hat{C}_n^B(\mathbf{u})$. Its asymptotic behavior can be established by means of an asymptotic result on the block bootstrap for general d -dimensional distribution functions established by Bühlmann [7, Theorem 3.1], see also Example 2.10 in Bücher and Volgushev [6].

Theorem 2. *Consider observations $\mathbf{X}_1, \dots, \mathbf{X}_n$, drawn from a strictly stationary process $(\mathbf{X}_j)_{j \in \mathbb{Z}}$ satisfying $\sum_{r=1}^{\infty} (r+1)^{16(d+1)} \sqrt{\alpha_{\mathbf{X}}(r)} < \infty$. Assume that $l_B(n) = \mathcal{O}(n^{1/2-\varepsilon})$ for $0 < \varepsilon < 1/2$. If C satisfies Condition (3), then the block bootstrap empirical copula process converges weakly conditional on $\mathbf{X}_1, \dots, \mathbf{X}_n$ in probability in $(\ell^\infty([0, 1]^d), \|\cdot\|_\infty)$:*

$$\mathbb{G}_{C,n}^B(\mathbf{u}) = \sqrt{n} \left\{ \hat{C}_n^B(\mathbf{u}) - \hat{C}_n(\mathbf{u}) \right\} \xrightarrow[H]{\mathbb{P}} \mathbb{G}_C(\mathbf{u}).$$

A brief remark on the condition on the mixing rate is in order here. The summability conditions forces $\alpha_{\mathbf{X}}(r)$ to be of order $O(r^{-96-\varepsilon})$ for dimension $d = 2$, which is far away from the (sharp) rate $O(r^{-1-\varepsilon})$ needed in Theorem 1. This discrepancy is due to the fact that the literature does not provide stronger results on the consistency of the block bootstrap for the d -variate empirical process except the ones in [7], at least to the best of our knowledge. Exploiting more recent techniques (which are beyond of the scope of this paper) we believe that it is possible to get better rates which are comparable to those of the non-bootstrap version, see also [35] and [37]. The proof of Theorem 2 being based on the functional delta method would easily transfer these rates to the rank-based copula setting considered in the present paper. Also note that for applications the message is not too bad: most time series models have exponentially decreasing alpha mixing coefficients, see Example 1 or more precisely the examples in [6].

2.2.2. The multiplier bootstrap

A process related to the bootstrap empirical copula process defined in (7) can be formulated if both the assumption of multinomially distributed random variables is dropped and the marginal distribution functions are left unaltered during the resampling procedure. The resulting bootstrap scheme is known for the i.i.d. context as the *multiplier bootstrap* or the *multiplier method*, see [45, 40, 4, 47]. In the present section we briefly summarize this concept and extend it to the serially dependent case.

Consider i.i.d. multiplier random variables ξ_1, \dots, ξ_n with mean and variance 1, additionally satisfying $\|\xi_j\|_{2,1} := \int_0^\infty \sqrt{P(|\xi_j| > x)} dx < \infty$ for all $j = 1, \dots, n$ (where the last condition is slightly stronger than that of a finite second moment). Replacing the multinomial multiplier random variables W_1, \dots, W_n by $\xi_1/\bar{\xi}, \dots, \xi_n/\bar{\xi}$, where $\bar{\xi} = n^{-1} \sum_{j=1}^n \xi_j$, (ensuring realizations having arithmetic mean one) yields the *multiplier (empirical copula) process* which converges weakly conditional on $\mathbf{X}_1, \dots, \mathbf{X}_n$ in probability in $(\ell^\infty([0, 1]^d), \|\cdot\|_\infty)$ in the i.i.d. situation, see Bücher and Dette [4] and more precisely in Bücher et al. [5, Theorem 2.3]:

$$\widehat{\mathbb{B}}_{C,n}^\xi(\mathbf{u}) = \sqrt{n} \left\{ \frac{1}{n} \sum_{j=1}^n \frac{\xi_j}{\bar{\xi}} \mathbf{1}_{\{\widehat{\mathbf{U}}_j \leq \mathbf{u}\}} - \widehat{C}_n(\mathbf{u}) \right\} \xrightarrow[\xi]{\mathbb{P}} \mathbb{B}_C(\mathbf{u}).$$

For general considerations of multiplier empirical processes, we refer to the monographs of van der Vaart and Wellner [50] and Kosorok [29]. The process is introduced by Scaillet [45] in a bivariate context, a general multivariate version and its unconditional weak convergence are investigated by Rémillard and Scaillet [40] and Segers [47]. In order to get approximations of the limiting process \mathbb{G}_C these authors propose to estimate the partial derivatives of the copula in (6) by some estimator $\widehat{D_i C_n}$ (see, e.g., Example 2 below) and define

$$\widehat{\mathbb{G}}_{C,n}^\xi(\mathbf{u}) = \widehat{\mathbb{B}}_{C,n}^\xi(\mathbf{u}) - \sum_{i=1}^d \widehat{D_i C_n}(\mathbf{u}) \widehat{\mathbb{B}}_{C,n}^\xi(\mathbf{u}^{(i)}),$$

which converges to \mathbb{G}_C conditionally on the data in probability, see [4, 3]. Here, the estimator $\widehat{D_i C_n}$ is supposed to satisfy the following two assumptions, which will also be necessary in the serially dependent situation:

C1 There exists a constant K such that $\|\widehat{D_i C_n}\|_\infty \leq K$ for all $n \in \mathbb{N}$.

C2 For all $\delta \in (0, 1/2)$ one has

$$\sup_{\mathbf{u} \in [0,1]^d: u_i \in [\delta, 1-\delta]} \left| \widehat{D_i C_n}(\mathbf{u}) - D_i C(\mathbf{u}) \right| \xrightarrow{\mathbb{P}} 0.$$

Example 2. It is easily seen that finite differencing yields a simple nonparametric estimator for the first order partial derivatives $D_i C(\mathbf{u})$ which satisfies **C1** and **C2**. More precisely, we define

$$\widehat{D_i C}(\mathbf{u}) = \begin{cases} \frac{\widehat{C}_n(\mathbf{u} + h\mathbf{e}_i) - \widehat{C}_n(\mathbf{u} - h\mathbf{e}_i)}{2h} & \text{for all } u \in [0, 1]^d, \ h \leq u_i \leq 1 - h, \\ \frac{\widehat{C}_n(\mathbf{u} + 2h\mathbf{e}_i)}{2h} & \text{for all } u \in [0, 1]^d, \ 0 \leq u_i < h, \\ \frac{\widehat{C}_n(\mathbf{u}) - \widehat{C}_n(\mathbf{u} - 2h\mathbf{e}_i)}{2h} & \text{for all } u \in [0, 1]^d, \ 1 - h < u_i \leq 1, \end{cases} \quad (10)$$

where $h = h_n \rightarrow 0$ such that $\inf_n h_n \sqrt{n} > 0$ and where \mathbf{e}_i denotes the i th column of the $d \times d$ identity matrix [see 47, 3].

Bücher and Dette [4] find that the multiplier technique yields more precise results than the nonparametric bootstrap in mean as well as in mean squared error when estimating the asymptotic covariance of the empirical copula process in the i.i.d. context. Motivated by the fact that this technique is inconsistent when applied to serially dependent samples,

we derive a generalization of the multiplier technique in the following. Inoue [26] develops a block multiplier process for general distribution functions based on dependent data in which the same multiplier random variable is used for a block of observations and the composition of blocks remains unaltered throughout the procedure. We consider a technique in which the composition of blocks is not fixed. More precisely, a *tapered block multiplier (empirical copula) process* is introduced based on the work of Bühlmann [7, Chapter 3.3] and Paparoditis and Politis [31]: The main idea is to consider a sample $\xi_{1,n}, \dots, \xi_{n,n}$ from a process $(\xi_{j,n})_{j \in \mathbb{Z}}$ of serially dependent *tapered block multiplier random variables*, satisfying:

A1 $(\xi_{j,n})_{j \in \mathbb{Z}}$ is independent of the observation process $(\mathbf{X}_j)_{j \in \mathbb{Z}}$.

A2 $(\xi_{j,n})_{j \in \mathbb{Z}}$ is a positive $c \cdot l(n)$ -dependent process, i.e., for fixed $j \in \mathbb{Z}$, $\xi_{j,n}$ is independent of $\xi_{j+h,n}$ for all $|h| \geq c \cdot l(n)$, where c is a constant and $l(n) \rightarrow \infty$ as $n \rightarrow \infty$ while $l(n) = o(n)$.

A3 $(\xi_{j,n})_{j \in \mathbb{Z}}$ is strictly stationary. For all $j, h \in \mathbb{Z}$, assume $E[\xi_{j,n}] = \mu > 0$, $\text{Cov}[\xi_{j,n}, \xi_{j+h,n}] = \mu^2 v(h/l(n))$ and v is a bounded function symmetric about zero; without loss of generality, we consider $\mu = 1$ and $v(0) = 1$. All central moments of $\xi_{j,n}$ are supposed to be bounded.

The following theorem is the main result of this section. Weak convergence of the tapered block multiplier process conditional on a sample $\mathbf{X}_1, \dots, \mathbf{X}_n$ is established. Regarding the strong assumptions on the mixing rate the remark after Theorem 2 holds true here as well.

Theorem 3. Consider observations $\mathbf{X}_1, \dots, \mathbf{X}_n$ drawn from a strictly stationary process $(\mathbf{X}_j)_{j \in \mathbb{Z}}$ satisfying $\sum_{r=1}^{\infty} (r+1)^c \sqrt{\alpha_{\mathbf{X}}(r)} < \infty$, where $c = \max\{8d+12, \lfloor 2/\varepsilon \rfloor + 1\}$. Let the tapered block multiplier process $(\xi_{j,n})_{j \in \mathbb{Z}}$ satisfy **A1**, **A2**, **A3** with block length $l(n) \rightarrow \infty$, where $l(n) = \mathcal{O}(n^{1/2-\varepsilon})$ for $0 < \varepsilon < 1/2$. Then,

$$\widehat{\mathbb{B}}_{C,n}^M(\mathbf{u}) = \sqrt{n} \left(\frac{1}{n} \sum_{j=1}^n \frac{\xi_{j,n}}{\bar{\xi}_n} \mathbf{1}_{\{\widehat{\mathbf{U}}_j \leq \mathbf{u}\}} - \widehat{C}_n(\mathbf{u}) \right) \xrightarrow[\xi]{\mathbb{P}} \mathbb{B}_C(\mathbf{u}) \quad \text{in } \ell^\infty([0, 1]^d),$$

where $\bar{\xi}_n = n^{-1} \sum_{j=1}^n \xi_{j,n}$. Moreover, if (3) holds and if $\widehat{D_i C_n}$ satisfies conditions **C1** and **C2**, then

$$\widehat{\mathbb{G}}_{C,n}^M(\mathbf{u}) = \widehat{\mathbb{B}}_{C,n}^M(\mathbf{u}) - \sum_{i=1}^d \widehat{D_i C_n}(\mathbf{u}) \widehat{\mathbb{B}}_{C,n}^M(\mathbf{u}^{(i)}) \xrightarrow[\xi]{\mathbb{P}} \mathbb{G}_C(\mathbf{u}) \quad \text{in } \ell^\infty([0, 1]^d).$$

Remark 1. The multiplier random variables can as well be assumed to be centered around zero [cf. 29, Proof of Theorem 2.6]. Define $\xi_{j,n}^0 := \xi_{j,n} - \mu$. Then

$$\widehat{\mathbb{B}}_{C,n}^{M,0}(\mathbf{u}) = \frac{1}{\sqrt{n}} \sum_{j=1}^n (\xi_{j,n}^0 - \bar{\xi}_n^0) \mathbf{1}_{\{\widehat{\mathbf{U}}_j \leq \mathbf{u}\}} = \bar{\xi}_n \frac{1}{\sqrt{n}} \sum_{j=1}^n \left(\frac{\xi_{j,n}}{\bar{\xi}_n} - 1 \right) \mathbf{1}_{\{\widehat{\mathbf{U}}_j \leq \mathbf{u}\}} = \bar{\xi}_n \widehat{\mathbb{B}}_{C,n}^M(\mathbf{u})$$

for all $\mathbf{u} \in [0, 1]^d$. This is an asymptotically equivalent form of the above tapered block multiplier process:

$$\sup_{[0,1]^d} \left| \widehat{\mathbb{B}}_{C,n}^{M,0}(\mathbf{u}) - \widehat{\mathbb{B}}_{C,n}^M(\mathbf{u}) \right| = \sup_{[0,1]^d} \left| (\bar{\xi}_n - 1) \widehat{\mathbb{B}}_{C,n}^M(\mathbf{u}) \right| \rightarrow 0$$

almost surely, since $\widehat{\mathbb{B}}_{C,n}^M(\mathbf{u})$ tends to a tight centered Gaussian limit, unconditionally. The assumption of centered multiplier random variables is abbreviated as **A3b** in the following.

In practice, given observations $\mathbf{X}_1, \dots, \mathbf{X}_n$ of a strictly stationary process $(\mathbf{X}_j)_{j \in \mathbb{Z}}$ satisfying the assumptions of Theorem 3, approximating \mathbb{G}_C requires the following three steps:

1. Estimate the partial derivatives of C by an estimator satisfying conditions **C1** and **C2**, for instance by the estimator given in Example 2 above.
2. For $s = 1, \dots, S$ with $S \in \mathbb{N}$, simulate samples $\xi_{1,n}^{(s)}, \dots, \xi_{n,n}^{(s)}$ from a tapered block multiplier process $(\xi_{j,n})_{j \in \mathbb{Z}}$ satisfying **A1**, **A2**, **A3**. For each $s \in \{1, \dots, S\}$, calculate

$$\widehat{\mathbb{B}}_{C,n}^{M(s)}(\mathbf{u}) = \frac{1}{\sqrt{n}} \sum_{j=1}^n \left(\frac{\xi_{j,n}^{(s)}}{\bar{\xi}_n^{(s)}} - 1 \right) \mathbf{1}_{\{\widehat{\mathbf{U}}_j \leq \mathbf{u}\}} \text{ for all } \mathbf{u} \in [0, 1]^d. \quad (11)$$

3. For each $s \in \{1, \dots, S\}$, calculate

$$\widehat{\mathbb{G}}_{C,n}^{M(s)}(\mathbf{u}) = \widehat{\mathbb{B}}_{C,n}^{M(s)}(\mathbf{u}) - \sum_{i=1}^d \widehat{D_i C_n}(\mathbf{u}) \widehat{\mathbb{B}}_{C,n}^{M(s)}(\mathbf{u}^{(i)}) \text{ for all } \mathbf{u} \in [0, 1]^d. \quad (12)$$

The sample $\{\widehat{\mathbb{G}}_{C,n}^{M(s)}\}_{s=1, \dots, S}$ is an approximate sample of \mathbb{G}_C .

There are numerous ways to define tapered block multiplier processes $(\xi_{j,n})_{j \in \mathbb{Z}}$ satisfying the assumptions **A1**, **A2**, **A3** made in Theorem 3. In the remaining part of this section, a basic version having uniform weights and a refined version with triangular weights are investigated and compared.

Example 3. A simple form of the tapered block multiplier random variables can be defined based on moving average processes. Consider the function κ_1 which assigns uniform weights given by

$$\kappa_1(h) := \begin{cases} \frac{1}{2l(n)-1} & \text{for all } |h| < l(n) \\ 0 & \text{else.} \end{cases}$$

Note that κ_1 is a discrete kernel, i.e., it is symmetric about zero and $\sum_{h \in \mathbb{Z}} \kappa_1(h) = 1$. The tapered block multiplier process is defined by

$$\xi_j = \xi_{j,n} = \sum_{h=-\infty}^{\infty} \kappa_1(h) w_{j+h} \text{ for all } j \in \mathbb{Z}, \quad (13)$$

where $(w_j)_{j \in \mathbb{Z}}$ is an independent and identically distributed sequence of, e.g., Gamma(q, q) random variables with $q := 1/[2l(n)-1]$. The expectation of ξ_j is then given by $E[\xi_j] = 1$, its variance by $\text{Var}[\xi_j] = 1$ for all $j \in \mathbb{Z}$. For all $j \in \mathbb{Z}$ and $|h| < 2l(n) - 1$, direct calculations further yield the covariance function $\text{Cov}(\xi_j, \xi_{j+h}) = \{2l(n) - 1 - |h|\} / \{2l(n) - 1\}$ which linearly decreases as h increases in absolute value. The resulting sequence $(\xi_j)_{j \in \mathbb{Z}}$ satisfies **A1**, **A2**, and **A3**. Exploring Remark 1, tapered block multiplier random variables can as well be defined based on sequences $(w_j)_{j \in \mathbb{Z}}$ of, e.g., Rademacher-type random variables w_j characterized by $P(w_j = -1/\sqrt{q}) = P(w_j = 1/\sqrt{q}) = 0.5$ or Normal random variables $w_j \sim N(0, 1/\sqrt{q})$. In either one of these two cases, the resulting sequence $(\xi_j)_{j \in \mathbb{Z}}$ satisfies **A1**, **A2**, and **A3b**. Figure 1 shows the kernel function κ_1 and simulated trajectories of Rademacher-type tapered block multiplier random variables.

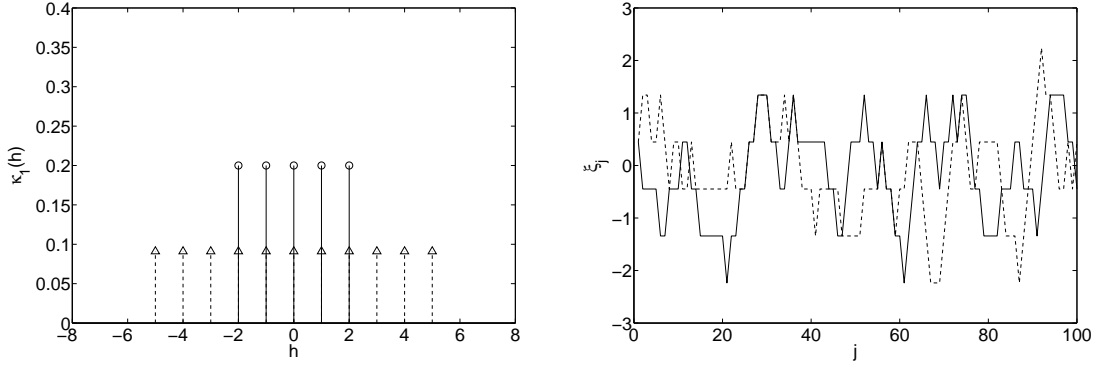


Figure 1: **Tapered block multiplier Monte Carlo simulation.** Kernel function $\kappa_1(h)$ (left) and simulated trajectories of Rademacher-type tapered block multiplier random variables ξ_1, \dots, ξ_{100} (right) with block length $l(n) = 3$ (solid line) and $l(n) = 6$ (dashed line), respectively.

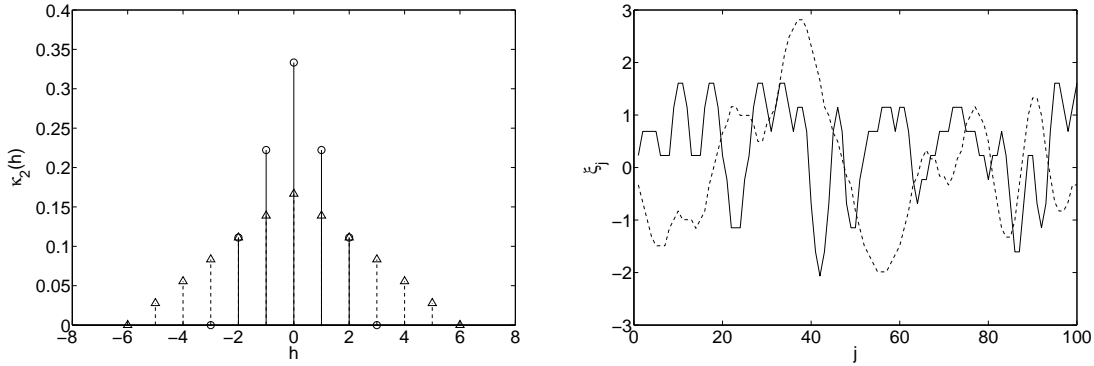


Figure 2: **Tapered block multiplier Monte Carlo simulation.** Kernel function $\kappa_2(h)$ (left) and simulated trajectories of Rademacher-type tapered block multiplier random variables ξ_1, \dots, ξ_{100} (right) with block length $l(n) = 3$ (solid line) and $l(n) = 6$ (dashed line), respectively.

Example 4. Following Bühlmann [7], let us define the kernel function by

$$\kappa_2(h) := \max\{0, \{1 - |h|/l(n)\}/l(n)\}$$

for all $h \in \mathbb{Z}$. The tapered multiplier process $(\xi_j)_{j \in \mathbb{Z}}$ follows Equation (13), where $(w_j)_{j \in \mathbb{Z}}$ is an independent and identically distributed sequence of $\text{Gamma}(q, q)$ random variables with $q = 2/\{3l(n)\} + 1/\{3l(n)^3\}$. The expectation of ξ_j is given by

$$E[\xi_j] = \frac{1}{l(n)} + 2 \sum_{h=1}^{l(n)} \frac{1}{l(n)} \left\{ 1 - \frac{h}{l(n)} \right\} = 1.$$

For the variance, direct calculations yield

$$\text{Var}[\xi_j] = \left[\frac{1}{l(n)^2} + 2 \sum_{h=1}^{l(n)} \frac{\{l(n) - h\}^2}{l(n)^4} \right] \text{Var}[w.] = \left\{ \frac{2}{3l(n)} + \frac{1}{3l(n)^3} \right\} \text{Var}[w.] = 1$$

for all $j \in \mathbb{Z}$. For any $j \in \mathbb{Z}$ and $|h| < 2l(n) - 1$, the covariance function $\text{Cov}(\xi_j, \xi_{j+h})$ can be described by a parabola centered at zero and opening downward [for details, see 7, Section 6.2]. The resulting sequence $(\xi_j)_{j \in \mathbb{Z}}$ satisfies **A1**, **A2**, and **A3**. Figure 2 provides an illustration of the kernel function κ_2 as well as simulated trajectories of Rademacher-type

tapered block multiplier random variables. In this setting, $(\xi_j)_{j \in \mathbb{Z}}$ satisfies **A1**, **A2**, and **A3b**. Notice the smoothing which is driven by the choice of kernel function and the block length $l(n)$. This effect can be further explored using more sophisticated kernel functions, e.g., with bell-shape; this is left for further research.

2.3. Finite sample behavior

In the present section we investigate and compare the finite sample properties of the (moving) block bootstrap and the tapered block multiplier technique by means of a simulation study. For that purpose we use both resampling techniques in order to estimate the (co)variances of the empirical copula process. This study complements the one of Bücher and Dette [4] and Bücher [3] on bootstrap approximations for the empirical copula process in the i.i.d setting.

To be more precise, in Tables 1, 2 and 3 we demonstrate simulation results (for $n = 100$ and $n = 200$) on the estimation of the theoretical covariance $\text{Cov}(\mathbb{G}_C(\mathbf{u}), \mathbb{G}_C(\mathbf{v}))$ calculated at the points $\mathbf{u} = \mathbf{v} \in \{(1/3, 1/3), (2/3, 1/3), (1/3, 2/3), (2/3, 2/3)\}$ in basically six different models: serial dependence features arise from (bivariate) i.i.d., AR(1) or GARCH(1, 1) time series models, while the copulas linking the marginals of the innovations are taken from the Gumbel–Hougaard or the Clayton family. The technical details of the execution of the study are given below, we start with the discussion of the results.

The results based on independent and identically distributed samples indicate that the tapered block multiplier outperforms the block bootstrap in mean and MSE of estimation. However, applying the resampling methods of the present paper in an i.i.d. setting comes at the price of an slight increased mean squared error in comparison to the multiplier or (block) bootstrap with block length $l = 1$ as investigated in [4]. Hence, we suggest to test serial independence of continuous multivariate time-series as introduced by Kojadinovic and Yan [28] to investigate which method is appropriate. In the case of serially dependent observations, the results indicate that the tapered block multiplier yields more precise results in mean and mean squared error than the block bootstrap (which tends to overestimate) for the considered choices of the temporal dependence structure, the kernel function and the copula. This finding coincides with the results in [4]. Regarding the choice of the kernel function, mean results for κ_1 and κ_2 are similar, whereas κ_2 yields slightly better results in mean squared error. Additional MC simulations are given in Ruppert [43]: if the multiplier or bootstrap methods for independent observations are incorrectly applied to dependent observations, i.e., $l_B = l_M = 1$, then their results do not reflect the changed structure adequately. Results based on Normal, Gamma, and Rademacher-type sequences $(w_j)_{j \in \mathbb{Z}}$ indicate that different distributions used to simulate the multiplier random variables lead to similar results. To ease comparison of the next section with the work of Rémillard and Scaillet [40], we decided to state the results for normal multiplier random variables here.

Finally, for the sake of completeness we state the technical details underlying the simulation. First of all, the details regarding the models are as follows.

- The bivariate Clayton and Gumbel–Hougaard copulas are given by

$$C_{\theta}^{Cl}(u_1, u_2) = \left(u_1^{-\theta} + u_2^{-\theta} - 1\right)^{-\frac{1}{\theta}}, \quad \theta > 0,$$

$$C_{\theta}^{Gu}(u_1, u_2) = \exp\left(-\left[\{-\ln(u_1)\}^{\theta} + \{-\ln(u_2)\}^{\theta}\right]^{\frac{1}{\theta}}\right), \quad \theta \geq 1,$$

Table 1: **Mean and MSE ($\times 10^4$) Monte Carlo results.** I.i.d. and AR(1) settings, sample size $n = 100$ and 1,000 Monte Carlo replications. For each replication, we perform $S = 2,000$ tapered block multiplier (M_i) repetitions with Normal multiplier random variables, kernel function κ_i , $i = 1, 2$, block length $l_M = 3$, and block bootstrap (B) repetitions with block length $l_B = 5$.

(u_1, u_2)		(1/3, 1/3)		(1/3, 2/3)		(2/3, 1/3)		(2/3, 2/3)	
		Mean	MSE	Mean	MSE	Mean	MSE	Mean	MSE
i.i.d. setting									
Clayton ($\theta = 1$)	True	0.0486		0.0338		0.0338		0.0508	
	Approx.	0.0487		0.0338		0.0338		0.0508	
	M_2	0.0496	1.6323	0.0344	1.2449	0.0345	1.3456	0.0528	1.6220
	M_1	0.0494	1.7949	0.0342	1.2934	0.0343	1.4128	0.0524	1.7910
	B	0.0599	2.8286	0.0432	2.4103	0.0429	2.1375	0.0643	3.1538
Clayton ($\theta = 4$)	True	0.0254		0.0042		0.0042		0.0389	
	Approx.	0.0255		0.0042		0.0042		0.0390	
	M_2	0.0259	0.9785	0.0051	0.3715	0.0048	0.3662	0.0407	1.6324
	M_1	0.0257	1.0104	0.0050	0.3656	0.0048	0.3673	0.0404	1.7048
	B	0.0383	2.5207	0.0100	0.7222	0.0097	0.6662	0.0533	3.7255
Gumbel ($\theta = 1.5$)	True	0.0493		0.0336		0.0336		0.0484	
	Approx.	0.0493		0.0335		0.0335		0.0485	
	M_2	0.0514	1.3914	0.0346	1.2657	0.0340	1.2583	0.0497	1.4946
	M_1	0.0510	1.5530	0.0344	1.3390	0.0338	1.3275	0.0495	1.6948
	B	0.0616	2.8334	0.0429	2.1366	0.0432	2.2273	0.0620	3.2134
Gumbel ($\theta = 3$)	True	0.0336		0.0058		0.0058		0.0293	
	Approx.	0.0335		0.0058		0.0058		0.0294	
	M_2	0.0355	1.1359	0.0064	0.4427	0.0063	0.3851	0.0307	0.9819
	M_1	0.0353	1.1991	0.0063	0.4409	0.0062	0.3832	0.0306	1.0261
	B	0.0470	2.7885	0.0120	0.8396	0.0122	0.8959	0.0437	2.9355
AR(1) setting with $\beta = 0.5$									
Clayton ($\theta = 1$)	Approx.	0.0599		0.0408		0.0409		0.0629	
	M_2	0.0602	3.1797	0.0394	2.4919	0.0398	2.5903	0.0625	2.6297
	M_1	0.0598	3.5305	0.0391	2.6090	0.0396	2.7410	0.0620	2.9826
	B	0.0699	3.4783	0.0496	2.9893	0.0492	2.8473	0.0761	4.0660
Clayton ($\theta = 4$)	Approx.	0.0329		0.0064		0.0064		0.0432	
	M_2	0.0347	2.0017	0.0071	0.6040	0.0072	0.6666	0.0460	2.8656
	M_1	0.0344	2.0672	0.0071	0.5869	0.0072	0.6583	0.0458	3.0571
	B	0.0472	3.5289	0.0132	1.0990	0.0128	0.9658	0.0585	4.2965
Gumbel ($\theta = 1.5$)	Approx.	0.0617		0.0408		0.0409		0.0605	
	M_2	0.0631	3.0587	0.0406	2.7512	0.0398	2.4187	0.0600	3.0433
	M_1	0.0626	3.4252	0.0402	2.8739	0.0395	2.4920	0.0594	3.3824
	B	0.0735	3.7410	0.0501	3.2025	0.0502	3.1150	0.0735	4.0521
Gumbel ($\theta = 3$)	Approx.	0.0385		0.0061		0.0061		0.0345	
	M_2	0.0425	2.5441	0.0069	0.5841	0.0070	0.5796	0.0375	2.1518
	M_1	0.0422	2.7239	0.0069	0.5706	0.0070	0.5836	0.0373	2.2381
	B	0.0535	3.8803	0.0127	1.0720	0.0125	1.0526	0.0507	4.1894

Table 2: **Mean and MSE ($\times 10^4$) Monte Carlo results.** I.i.d. and AR(1) settings, sample size $n = 200$ and 1,000 Monte Carlo replications. For each replication, we perform $S = 2,000$ tapered block multiplier (M_i) repetitions with Normal multiplier random variables, kernel function κ_i , $i = 1, 2$, block length $l_M = 4$, and block bootstrap (B) repetitions with block length $l_B = 7$.

(u_1, u_2)		(1/3, 1/3)		(1/3, 2/3)		(2/3, 1/3)		(2/3, 2/3)	
		Mean	MSE	Mean	MSE	Mean	MSE	Mean	MSE
i.i.d. setting									
Clayton ($\theta = 1$)	True	0.0486		0.0338		0.0338		0.0508	
	Approx.	0.0487		0.0338		0.0338		0.0508	
	M_2	0.0494	1.2579	0.0346	0.8432	0.0343	0.8423	0.0522	1.0987
	M_1	0.0490	1.3749	0.0345	0.8880	0.0341	0.8719	0.0519	1.2074
	B	0.0562	1.7155	0.0402	1.2154	0.0395	1.2279	0.0593	1.8137
Clayton ($\theta = 4$)	True	0.0254		0.0042		0.0042		0.0389	
	Approx.	0.0255		0.0042		0.0042		0.0390	
	M_2	0.0261	0.5811	0.0044	0.1889	0.0048	0.2027	0.0390	0.9932
	M_1	0.0260	0.6024	0.0044	0.1876	0.0048	0.2008	0.0388	1.0702
	B	0.0347	1.6381	0.0076	0.3197	0.0073	0.2908	0.0489	2.1425
Gumbel ($\theta = 1.5$)	True	0.0493		0.0336		0.0336		0.0484	
	Approx.	0.0493		0.0335		0.0335		0.0485	
	M_2	0.0512	1.1513	0.0347	0.8157	0.0347	0.8113	0.0504	1.1697
	M_1	0.0511	1.2909	0.0345	0.8653	0.0346	0.8692	0.0503	1.2811
	B	0.0577	1.7178	0.0402	1.2233	0.0403	1.1701	0.0574	1.8245
Gumbel ($\theta = 3$)	True	0.0336		0.0058		0.0058		0.0293	
	Approx.	0.0335		0.0058		0.0058		0.0294	
	M_2	0.0361	0.8340	0.0067	0.2577	0.0063	0.2505	0.0320	0.8678
	M_1	0.0359	0.9107	0.0067	0.2576	0.0062	0.2444	0.0318	0.9078
	B	0.0435	1.7448	0.0095	0.3917	0.0094	0.3804	0.0388	1.5320
AR(1) setting with $\beta = 0.5$									
Clayton ($\theta = 1$)	Approx.	0.0599		0.0408		0.0409		0.0629	
	M_2	0.0615	2.3213	0.0413	1.8999	0.0414	1.9235	0.0646	2.2534
	M_1	0.0608	2.5553	0.0408	1.9594	0.0410	2.0094	0.0638	2.4889
	B	0.0676	2.6206	0.0468	1.9802	0.0472	2.0278	0.0715	2.5494
Clayton ($\theta = 4$)	Approx.	0.0329		0.0064		0.0064		0.0432	
	M_2	0.0344	1.2685	0.0074	0.3890	0.0073	0.4108	0.0460	1.7936
	M_1	0.0341	1.3034	0.0073	0.3824	0.0072	0.4053	0.0455	1.8566
	B	0.0432	2.1693	0.0105	0.5427	0.0101	0.4901	0.0546	2.8808
Gumbel ($\theta = 1.5$)	Approx.	0.0617		0.0408		0.0409		0.0605	
	M_2	0.0649	2.2833	0.0432	2.1528	0.0433	1.9800	0.0646	3.0536
	M_1	0.0639	2.4663	0.0426	2.1943	0.0428	2.0367	0.0638	3.2590
	B	0.0703	2.5133	0.0468	1.8163	0.0467	1.7087	0.0693	2.5655
Gumbel ($\theta = 3$)	Approx.	0.0385		0.0061		0.0061		0.0345	
	M_2	0.0430	1.9156	0.0079	0.4884	0.0074	0.3950	0.0405	2.5846
	M_1	0.0426	1.9784	0.0078	0.4806	0.0073	0.3858	0.0400	2.6083
	B	0.0492	2.2152	0.0102	0.5375	0.0099	0.4832	0.0455	2.1534

Table 3: **Mean and MSE ($\times 10^4$) Monte Carlo results.** AR(1) and GARCH(1,1) settings, sample size $n = 200$ and 1,000 Monte Carlo replications. For each replication, we perform $S = 2,000$ tapered block multiplier (M_i) repetitions with Normal multiplier random variables, kernel function κ_i , $i = 1, 2$, block length $l_M = 4$, and block bootstrap (B) repetitions with block length $l_B = 7$.

(u_1, u_2)		(1/3, 1/3)		(1/3, 2/3)		(2/3, 1/3)		(2/3, 2/3)	
		Mean	MSE	Mean	MSE	Mean	MSE	Mean	MSE
AR(1) setting with $\beta = 0.25$									
Clayton ($\theta = 1$)	Approx.	0.0506		0.0350		0.0350		0.0530	
	M_2	0.0521	1.5319	0.0360	1.0527	0.0361	1.1258	0.0545	1.4689
	M_1	0.0517	1.6574	0.0357	1.1003	0.0358	1.1662	0.0541	1.6083
	B	0.0591	2.0394	0.0415	1.4084	0.0417	1.5256	0.0624	2.1027
Clayton ($\theta = 4$)	Approx.	0.0272		0.0048		0.0048		0.0395	
	M_2	0.0285	0.8366	0.0052	0.2290	0.0052	0.2208	0.0413	1.3315
	M_1	0.0283	0.8792	0.0051	0.2273	0.0052	0.2200	0.0410	1.3858
	B	0.0375	1.9937	0.0084	0.3726	0.0085	0.3646	0.0495	2.1996
Gumbel ($\theta = 1.5$)	Approx.	0.0518		0.0349		0.0348		0.0507	
	M_2	0.0549	1.4150	0.0373	1.1973	0.0377	1.2459	0.0550	2.0067
	M_1	0.0545	1.5154	0.0370	1.2339	0.0373	1.2871	0.0545	2.0766
	B	0.0608	2.0500	0.0415	1.3234	0.0418	1.4093	0.0608	2.3118
Gumbel ($\theta = 3$)	Approx.	0.0346		0.0058		0.0058		0.0304	
	M_2	0.0386	1.1686	0.0070	0.3079	0.0068	0.2979	0.0350	1.5135
	M_1	0.0384	1.2224	0.0070	0.3052	0.0067	0.2892	0.0347	1.5157
	B	0.0447	1.8434	0.0095	0.3873	0.0097	0.4303	0.0409	1.8797
GARCH(1,1) setting									
Clayton ($\theta = 1$)	Approx.	0.0479		0.0340		0.0340		0.0516	
	M_2	0.0491	1.1144	0.0347	0.8485	0.0343	0.8279	0.0520	1.0958
	M_1	0.0486	1.3579	0.0339	0.9021	0.0338	0.8100	0.0515	1.2013
	B	0.0567	2.0156	0.0403	1.1765	0.0403	1.2556	0.0600	1.8542
Clayton ($\theta = 4$)	Approx.	0.0252		0.0055		0.0056		0.0403	
	M_2	0.0259	0.5054	0.0051	0.1979	0.0053	0.2301	0.0399	1.0431
	M_1	0.0258	0.6429	0.0052	0.2199	0.0051	0.2217	0.0390	1.0959
	B	0.0345	1.5252	0.0081	0.2764	0.0081	0.2921	0.0484	1.8359
Gumbel ($\theta = 1.5$)	Approx.	0.0500		0.0339		0.0339		0.0482	
	M_2	0.0516	1.0480	0.0356	0.8486	0.0354	0.8175	0.0511	1.2451
	M_1	0.0516	1.2235	0.0351	0.9582	0.0352	0.8848	0.0503	1.2774
	B	0.0575	1.5928	0.0402	1.2395	0.0403	1.2346	0.0574	1.9198
Gumbel ($\theta = 3$)	Approx.	0.0341		0.0074		0.0074		0.0291	
	M_2	0.0362	0.9284	0.0073	0.2941	0.0072	0.2587	0.0321	0.9133
	M_1	0.0366	0.9782	0.0073	0.2786	0.0071	0.2888	0.0320	0.9819
	B	0.0435	1.6811	0.0103	0.3607	0.0101	0.3390	0.0390	1.6878

respectively, and we chose the parameters in such a way that Kendall's τ is either $1/3$ or $2/3$, i.e., $\theta \in \{1, 4\}$ for the Clayton and $\theta \in \{1.5, 3\}$ for the Gumbel–Hougaard copula.

- The AR(1) process we consider is the stationary solution of the AR(1) equation

$$\mathbf{X}_j = \beta \mathbf{X}_{j-1} + \boldsymbol{\varepsilon}_j,$$

where the innovations are supposed to have standard normal marginals linked by one of the aforementioned copulas and where the coefficient of the lagged variable is either $\beta = 0.25$ or $\beta = 0.5$. This stationary solution can be written as $\mathbf{X}_j = \sum_{k=0}^{\infty} \beta^k \boldsymbol{\varepsilon}_{j-k}$. We simulate an (approximate) sample of length n of this model as follows: for some reasonably large negative number M , e.g., $M = -100$, let $\mathbf{U}_j = (U_{j,1}, U_{j,2})$, $j = M, \dots, n$ be a sample of independent realizations of one of the aforementioned copulas. Set $\boldsymbol{\varepsilon}_j = (\Phi^{-1}(U_{j,1}), \Phi^{-1}(U_{j,2}))$, with Φ being the standard normal cdf, and recursively define $\mathbf{X}_M = \boldsymbol{\varepsilon}_M$ and

$$\mathbf{X}_j = \beta \mathbf{X}_{j-1} + \boldsymbol{\varepsilon}_j \text{ for all } j = M+1, \dots, n. \quad (14)$$

The last n observations form the sample $\mathbf{X}_1, \dots, \mathbf{X}_n$.

- The GARCH(1,1) sample is simulated as following: define $\boldsymbol{\varepsilon}_j = (\varepsilon_{j,1}, \varepsilon_{j,2})$ as in the AR(1)-example and recursively define $\sigma_{M,i} = \sqrt{\omega_i / (1 - \alpha_i - \beta_i)}$ for $i = 1, 2$ and

$$X_{j,i} = \sigma_{j,i} \varepsilon_{j,i}, \text{ with } \sigma_{j,i}^2 = \omega_i + \beta_i \sigma_{j-1,i}^2 + \alpha_i \varepsilon_{j-1,i}^2 \quad (15)$$

for $i = 1, 2$ and $j = M+1, \dots, n$, where $\omega_1 = 0.012, \omega_2 = 0.037, \beta_1 = 0.919, \beta_2 = 0.868, \alpha_1 = 0.072$ and $\alpha_2 = 0.115$. Again, the last n observations form the sample $\mathbf{X}_1, \dots, \mathbf{X}_n$. The considered coefficients are estimates derived in Jondeau et al. [27] to model volatility of S&P 500 and DAX daily (log-)returns in an empirical application which shows the practical relevance of this specific parameter choice.

The theoretical covariance of \mathbb{G}_C is easily calculated in the i.i.d. setting, but hardly derivable in the serially dependent case. Note that even a closed form expression for the copula of \mathbf{X}_j is unaccessible. For that reason, we approximate the theoretical covariance by means of empirical covariances calculated from simulated samples of the process $\sqrt{n}(\hat{C}_n - \hat{C}_N)$, where $n = n(N) \rightarrow \infty$ with $n = o(N)$. We chose $N = 10^6$ and $n = 1000$, and made 10^6 replications on basis of which we calculated the empirical covariances. The results in Tables 1 - 3 (the ‘True’ vs. ‘Approx.’ lines) for the i.i.d. setting show that this approximation works sufficiently well, whence we can use it as a benchmark in the two serial dependent settings.

Regarding the tapered block multiplier bootstrap we decided to use both kernel functions κ_1 and κ_2 from Examples 3 and 4. For the (moving) block bootstrap we choose the block length as $l_B(100) = 5$ and $l_B(200) = 7$; this choice corresponds to $l_B(n) = \lfloor 1.25n^{1/3} \rfloor$ which satisfies the assumptions of the asymptotic theory. For a detailed discussion on the block length of the block bootstrap, we refer to Künsch [30] as well as Bühlmann and Künsch [8]. The tapered block multiplier technique is assessed based on a sequence $(w_j)_{j \in \mathbb{Z}}$ of normal random variables as introduced in Examples 3 and 4. The block length is set to $l_M(n) = \lfloor 1.1n^{1/4} \rfloor$, hence $l_M(100) = 3$ and $l_M(200) = 4$, meaning that both methods yield $2l_M$ -dependent blocks. For all methods $S = 2,000$ bootstrap repetitions are performed and the target covariance is estimated by the sample covariance over the 2,000 repetitions. This procedure is repeated 1,000 times and we report mean and mean squared error (MSE) for each method.

3. Testing for a constant copula

The present section presents two nonparametric tests for a constant copula in the case of serially dependent processes. We begin with a test where a change point candidate is given, and proceed with a more general test without specifying a time point where a break in the copula structure occurs. Both tests are consistent against general alternatives and their finite sample performance is investigated by means of a simulation study.

3.1. Specified change point candidate

The specification of a change point candidate can for instance have an economic motivation: Patton [32] investigates a change in parameters of the dependence structure between various exchange rates following the introduction of the euro on the 1st of January 1999. Focusing on stock returns, multivariate association between major S&P global sector indices before and after the bankruptcy of Lehman Brothers Inc. on 15th of September 2008 is assessed in Gaißer et al. [21] and Ruppert [43]. Whereas these references investigate change points in functionals of the copula, the copula itself is in the focus of this study. This approach permits to analyze changes in the structure of association even if a functional thereof, such as a measure of multivariate association, is invariant.

Suppose we observe a sample $\mathbf{X}_1, \dots, \mathbf{X}_n$ of a process $(\mathbf{X}_j)_{j \in \mathbb{Z}}$. We derive a test for constancy of the copula in the case of a specified change point candidate indexed by $\lfloor \lambda n \rfloor$ for $\lambda \in (0, 1)$, i.e., a test for the hypothesis

$$\begin{aligned} H_0 : \mathbf{U}_j &\sim C_1 \text{ for all } j = 1, \dots, n, \\ H_1 : \mathbf{U}_j &\sim \begin{cases} C_1 & \text{for all } j = 1, \dots, \lfloor \lambda n \rfloor, \\ C_2 & \text{for all } j = \lfloor \lambda n \rfloor + 1, \dots, n, \end{cases} \end{aligned}$$

where C_1 and C_2 are assumed to be different in at least one point $\mathbf{u} \in [0, 1]^d$ (and hence, for continuity reasons, also in a neighbourhood of this point). The proposed test statistic will be based on a splitting of the sample into two subsamples: $\mathbf{X}_1, \dots, \mathbf{X}_{\lfloor \lambda n \rfloor}$ and $\mathbf{X}_{\lfloor \lambda n \rfloor + 1}, \dots, \mathbf{X}_n$. A significant discrepancy between estimates of the copula in the two subsamples suggests to reject the null hypothesis. Assuming constant marginal distributions in each subsample, let

$$\hat{C}_{1, \dots, \lfloor \lambda n \rfloor}(\mathbf{u}) = \frac{1}{\lfloor \lambda n \rfloor} \sum_{j=1}^{\lfloor \lambda n \rfloor} \mathbf{1}_{\{\hat{\mathbf{U}}_j \leq \mathbf{u}\}}, \quad \hat{C}_{\lfloor \lambda n \rfloor + 1, \dots, n}(\mathbf{u}) = \frac{1}{n - \lfloor \lambda n \rfloor} \sum_{j=\lfloor \lambda n \rfloor + 1}^n \mathbf{1}_{\{\hat{\mathbf{V}}_j \leq \mathbf{u}\}}$$

denote the corresponding empirical copulas of the two subsamples, where $\hat{\mathbf{U}}_1, \dots, \hat{\mathbf{U}}_{\lfloor \lambda n \rfloor}$ and $\hat{\mathbf{V}}_{\lfloor \lambda n \rfloor + 1}, \dots, \hat{\mathbf{V}}_n$ denote the pseudo-observations calculated from the first and second subsample, respectively. We consider the test statistic defined by

$$T_n(\lambda) = \int_{[0, 1]^d} \left[\sqrt{\frac{\lfloor \lambda n \rfloor (n - \lfloor \lambda n \rfloor)}{n}} \left\{ \hat{C}_{1, \dots, \lfloor \lambda n \rfloor}(\mathbf{u}) - \hat{C}_{\lfloor \lambda n \rfloor + 1, \dots, n}(\mathbf{u}) \right\} \right]^2 d\mathbf{u} \quad (16)$$

which can be calculated explicitly; for details we refer to Rémillard and Scaillet [40]. These authors introduce a test for equality between two copulas which is applicable in the case of serial independence. Weak convergence of $T_n(\lambda)$ under strong mixing follows from the following result.

Theorem 4. Consider observations $\mathbf{X}_1, \dots, \mathbf{X}_n$, drawn from a process $(\mathbf{X}_j)_{j \in \mathbb{Z}}$ satisfying the strong mixing condition $\alpha_{\mathbf{X}}(r) = \mathcal{O}(r^{-a})$ for some $a > 1$. Further assume a specified change point candidate indexed by $\lfloor \lambda n \rfloor$ for $\lambda \in (0, 1)$ such that $\mathbf{U}_j \sim C_1$, $X_{j,i} \sim F_{1,i}$ for all $j = 1, \dots, \lfloor \lambda n \rfloor$, $i = 1, \dots, d$ and $\mathbf{U}_j \sim C_2$, $X_{j,i} \sim F_{2,i}$ for all $j = \lfloor \lambda n \rfloor + 1, \dots, n$, $i = 1, \dots, d$. Suppose that C_1 and C_2 satisfy Condition (3). Under the null hypothesis $C_1 = C_2$, in the metric space $\ell^\infty([0, 1]^d)$,

$$\sqrt{\frac{\lfloor \lambda n \rfloor (n - \lfloor \lambda n \rfloor)}{n}} \left\{ \widehat{C}_{1, \dots, \lfloor \lambda n \rfloor} - \widehat{C}_{\lfloor \lambda n \rfloor + 1, \dots, n} \right\} \xrightarrow{w.} \sqrt{1 - \lambda} \mathbb{G}_{C_1} - \sqrt{\lambda} \mathbb{G}_{C_2},$$

where $\mathbb{G}_{C_p}(\mathbf{u}) = \mathbb{B}_{C_p}(\mathbf{u}) - \sum_{i=1}^d D_i C(\mathbf{u}) \mathbb{B}_{C_p}(\mathbf{u}^{(i)})$ for $p = 1, 2$ and where \mathbb{B}_{C_1} and \mathbb{B}_{C_2} denote two independent tight centered Gaussian processes with covariances as specified in (5).

As a consequence of this Theorem, the continuous mapping Theorem yields

$$T_n(\lambda) \xrightarrow{w.} T(\lambda) = \int_{[0, 1]^d} \left\{ \sqrt{1 - \lambda} \mathbb{G}_{C_1}(\mathbf{u}) - \sqrt{\lambda} \mathbb{G}_{C_2}(\mathbf{u}) \right\}^2 d\mathbf{u}$$

for all $\lambda \in (0, 1)$. Note that if there exists a subset $\mathcal{I} \in [0, 1]^d$ such that

$$\int_{\mathcal{I}} \left\{ \sqrt{1 - \lambda} C_1(\mathbf{u}) - \sqrt{\lambda} C_2(\mathbf{u}) \right\}^2 d\mathbf{u} > 0,$$

then $T_n(\lambda) \rightarrow \infty$ in probability under H_1 . To estimate p-values of the test statistic, we use the tapered block multiplier technique developed in Section 2. For that purpose, let $(\xi_{n,j})_{j \in \mathbb{Z}}$ be a sequence of multipliers satisfying **A1**, **A2**, **A3** and set

$$\begin{aligned} \widehat{\mathbb{B}}_{C,n,1}^M(\mathbf{u}) &= \frac{1}{\sqrt{\lfloor \lambda n \rfloor}} \sum_{j=1}^{\lfloor \lambda n \rfloor} \left(\frac{\xi_{n,j}}{\bar{\xi}_n^{(1)}} - 1 \right) \mathbf{1}_{\{\widehat{U}_j \leq \mathbf{u}\}} \\ \widehat{\mathbb{B}}_{C,n,2}^M(\mathbf{u}) &= \frac{1}{\sqrt{n - \lfloor \lambda n \rfloor}} \sum_{j=\lfloor \lambda n \rfloor + 1}^n \left(\frac{\xi_{n,j}}{\bar{\xi}_n^{(2)}} - 1 \right) \mathbf{1}_{\{\widehat{V}_j \leq \mathbf{u}\}}, \end{aligned}$$

where $\bar{\xi}_n^{(1)}$ and $\bar{\xi}_n^{(2)}$ denote the arithmetic mean of the $\xi_{j,n}$ in the corresponding samples. Let $\widehat{D_i C_{n,1}}$ and $\widehat{D_i C_{n,2}}$ denote estimators for the corresponding partial derivatives and set, for $p = 1, 2$,

$$\widehat{\mathbb{G}}_{C,n,p}^M(\mathbf{u}) = \widehat{\mathbb{B}}_{C,n,p}^M(\mathbf{u}) - \sum_{i=1}^d \widehat{D_i C_{n,p}}(\mathbf{u}) \widehat{\mathbb{B}}_{C,n,p}^M(\mathbf{u}^{(i)}).$$

In the i.i.d. multiplier case this statistic is, up to some negligible constants, the same as the one used in [40] to test for equality between two copulas.

Proposition 1. Consider observations $\mathbf{X}_1, \dots, \mathbf{X}_{\lfloor \lambda n \rfloor}$ and $\mathbf{X}_{\lfloor \lambda n \rfloor + 1}, \dots, \mathbf{X}_n$ drawn from a process $(\mathbf{X}_j)_{j \in \mathbb{Z}}$. Assume that the process satisfies the strong mixing assumptions of Theorem 3. Let $\lfloor \lambda n \rfloor$ for $\lambda \in (0, 1)$ denote a specific change point candidate such that $\mathbf{U}_j \sim C_1$, $X_{j,i} \sim F_{1,i}$ for all $j = 1, \dots, \lfloor \lambda n \rfloor$, $i = 1, \dots, d$ and $\mathbf{U}_j \sim C_2$, $X_{j,i} \sim F_{2,i}$ for all $j = \lfloor \lambda n \rfloor + 1, \dots, n$, $i = 1, \dots, d$. Suppose that C_1 and C_2 satisfy Condition (3) and that $\widehat{D_i C_{n,1}}$ and $\widehat{D_i C_{n,2}}$ are estimators for the corresponding partial derivatives satisfying **C1** and

C2. Let $\xi_{n,1}, \dots, \xi_{n,n}$ denote samples of a tapered block multiplier process $(\xi_{n,j})_{j \in \mathbb{Z}}$ satisfying **A1**, **A2**, **A3** with block length $l(n) \rightarrow \infty$, where $l(n) = \mathcal{O}(n^{1/2-\varepsilon})$ for $0 < \varepsilon < 1/2$. Then

$$\hat{\mathbb{H}}_{n,\lambda}^M := \sqrt{1-\lambda} \hat{\mathbb{G}}_{C,n,1}^M - \sqrt{\lambda} \hat{\mathbb{G}}_{C,n,2}^M \xrightarrow[\xi]{\mathbb{P}} \sqrt{1-\lambda} \mathbb{G}_{C_1} - \sqrt{\lambda} \mathbb{G}_{C_2} \quad (17)$$

in $\ell^\infty([0,1]^d)$ both under the null hypothesis as well as under the alternative.

As a consequence, by the continuous mapping Theorem for the bootstrap, see [29],

$$\hat{T}_n^M(\lambda) = \int_{[0,1]^d} (\hat{\mathbb{H}}_{n,\lambda}^M(\mathbf{u}))^2 d\mathbf{u} \xrightarrow[\xi]{\mathbb{P}} T(\lambda).$$

The integral involved in $\hat{T}_n^M(\lambda)$ can be calculated explicitly [see 39, Appendix B]. If we repeat the procedure S times to obtain a sample $\hat{T}_n^{M(1)}, \dots, \hat{T}_n^{M(S)}$, then an approximate p-value for the test for H_0 is provided by

$$\frac{1}{S} \sum_{s=1}^S \mathbf{1}_{\{\hat{T}_n^{M(s)}(\lambda) > T_n(\lambda)\}}. \quad (18)$$

Hence, p-values can be estimated by counting the number of cases in which the simulated test statistic based on the tapered block multiplier method exceeds the observed one.

Finite sample properties. Size and power of the test in finite samples are assessed in a simulation study. We consider bivariate samples of size $n = 100$ or $n = 200$ generated as in Section 2.3, i.e., marginal i.i.d., AR(1) and GARCH(1,1) processes are either linked by a Clayton or a Gumbel–Hougaard copula. The change point after observation $\lfloor \lambda n \rfloor = n/2$ only affects the parameter within each family: the copula C_1 is parameterized such that Kendall's $\tau_1 = 0.2$, the copula C_2 such that Kendall's $\tau_2 = 0.2, \dots, 0.9$. A set of $S = 2,000$ normal tapered block multiplier processes is simulated, where the kernel function is chosen as κ_2 as suggested by the simulation results in Section 2.3 and where $l_M(100) = 3$ and $l_M(200) = 4$ are chosen for the block length.

The results of 1,000 MC replications are shown in Tables 4 and 5 for $n = 100$ and $n = 200$, respectively. The test based on the tapered block multiplier technique leads to a rejection quota under the null hypothesis which is close to the chosen theoretical asymptotic size of 5% in all considered settings. Comparing the results for $n = 100$ and $n = 200$, we observe that the approximation of the asymptotic size based on the tapered block multiplier improves in precision with increased sample size. The tapered block multiplier-based test also performs well under the alternative hypothesis and its power increases with the difference $\tau_2 - \tau_1$ between the considered values for Kendall's τ . The power of the test under the alternative hypothesis is best in the case of no serial dependence as is shown in Table 5. If serial dependence is present in the sample then more observations are required to reach the power of the test in the case of serially independent observations. For comparison, we also show the results if the test assuming independent observations (i.e., the test based on the multiplier technique with block length $l = 1$) is erroneously applied to the simulated dependent observations. The effects of different types of dependent observations differ largely in the finite sample simulations considered: GARCH(1,1) processes do not show strong impact, whereas AR(1) processes lead to considerable distortions, in particular regarding the size of

Table 4: **Size and power of the test for a constant copula with a specified change point candidate.** Results are based on 1,000 Monte Carlo replications, $n = 100$, $S = 2,000$ tapered block multiplier repetitions, kernel function κ_2 , and asymptotic significance level $\alpha = 5\%$.

τ_2		0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
i.i.d. setting									
Clayton	$l = 1$	0.036	0.110	0.295	0.612	0.881	0.983	1.000	1.000
	$l = 3$	0.050	0.114	0.315	0.578	0.877	0.976	1.000	1.000
Gumbel	$l = 1$	0.040	0.093	0.236	0.569	0.840	0.976	0.998	1.000
	$l = 3$	0.063	0.110	0.276	0.594	0.866	0.983	1.000	1.000
GARCH(1,1) setting									
Clayton	$l = 1$	0.037	0.106	0.298	0.598	0.868	0.977	1.000	1.000
	$l = 3$	0.047	0.120	0.303	0.588	0.876	0.978	0.999	1.000
Gumbel	$l = 1$	0.043	0.089	0.246	0.573	0.827	0.978	0.999	1.000
	$l = 3$	0.065	0.124	0.285	0.569	0.847	0.980	1.000	1.000
AR(1) setting with $\beta = 0.25$									
Clayton	$l = 1$	0.051	0.115	0.308	0.592	0.849	0.969	0.999	1.000
	$l = 3$	0.047	0.111	0.292	0.547	0.836	0.968	0.998	1.000
Gumbel	$l = 1$	0.053	0.109	0.257	0.550	0.836	0.975	0.998	1.000
	$l = 3$	0.066	0.105	0.254	0.568	0.818	0.964	1.000	1.000
AR(1) setting with $\beta = 0.5$									
Clayton	$l = 1$	0.086	0.154	0.313	0.549	0.798	0.928	0.985	1.000
	$l = 3$	0.078	0.117	0.236	0.462	0.730	0.868	0.986	0.999
Gumbel	$l = 1$	0.100	0.172	0.285	0.541	0.816	0.956	0.998	1.000
	$l = 3$	0.077	0.109	0.218	0.482	0.722	0.907	0.994	0.999

the test. Results indicate that the test overrejects if temporal dependence is not taken into account; the observed size of the test in these cases can be more than twice the specified asymptotic size. For comparison, results for $n = 200$ and kernel function κ_1 are shown in Ruppert [43]. The obtained results indicate that the uniform kernel function κ_1 leads to a more conservative testing procedure since the rejection quota is slightly higher, both under the null hypothesis as well as under the alternative. Due to the fact that the size of the test is approximated more accurately based on the kernel function κ_2 , its use is recommended.

3.2. The general case: unspecified change point candidate

The assumption of a change point candidate at specified location is relaxed in the following. Intuitively, testing with unspecified change point candidate(s) is less restrictive but a trade-off is to be made: the tests introduced in this section neither require conditions on the partial derivatives of the underlying copula(s) nor the specification of change point candidate(s), yet they are based on the assumption of strictly stationary univariate processes, i.e., $X_{j,i} \sim F_i$ for all $j \in \mathbb{Z}$ and $i = 1, \dots, d$. The motivation for this test setting is that only for a subset of the change points documented in empirical studies, a priori hypothesis such as triggering economic events can be found [see, e.g., 15]. Even if a triggering event exists, its start (and end) often are subject to uncertainty: Rodriguez [42] studies changes in dependence structures of stock returns during periods of turmoil considering data framing the East Asian crisis in 1997 as well as the Mexican devaluation in 1994, where no change point candidate

Table 5: **Size and power of the test for a constant copula with a specified change point candidate.** Results are based on 1,000 Monte Carlo replications, $n = 200$, $S = 2,000$ tapered block multiplier repetitions, kernel function κ_2 , and asymptotic significance level $\alpha = 5\%$.

τ_2		0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
i.i.d. setting									
Clayton	$l = 1$	0.047	0.172	0.524	0.908	0.993	1.000	1.000	1.000
	$l = 4$	0.063	0.164	0.552	0.905	0.989	1.000	1.000	1.000
Gumbel	$l = 1$	0.043	0.162	0.525	0.877	0.991	1.000	1.000	1.000
	$l = 4$	0.055	0.169	0.535	0.895	0.996	1.000	1.000	1.000
GARCH(1,1) setting									
Clayton	$l = 1$	0.040	0.169	0.503	0.894	0.994	1.000	1.000	1.000
	$l = 4$	0.056	0.160	0.541	0.903	0.992	1.000	1.000	1.000
Gumbel	$l = 1$	0.046	0.154	0.498	0.873	0.992	1.000	1.000	1.000
	$l = 4$	0.057	0.174	0.496	0.899	0.994	1.000	1.000	1.000
AR(1) setting with $\beta = 0.25$									
Clayton	$l = 1$	0.052	0.180	0.521	0.866	0.989	1.000	1.000	1.000
	$l = 4$	0.057	0.149	0.497	0.867	0.988	1.000	1.000	1.000
Gumbel	$l = 1$	0.047	0.180	0.515	0.872	0.989	1.000	1.000	1.000
	$l = 4$	0.050	0.136	0.490	0.855	0.992	1.000	1.000	1.000
AR(1) setting with $\beta = 0.5$									
Clayton	$l = 1$	0.107	0.237	0.523	0.813	0.975	0.999	1.000	1.000
	$l = 4$	0.058	0.137	0.396	0.748	0.957	0.998	1.000	1.000
Gumbel	$l = 1$	0.122	0.227	0.499	0.825	0.979	1.000	1.000	1.000
	$l = 4$	0.059	0.123	0.401	0.770	0.958	0.998	1.000	1.000

is given a priori. These objects of investigation are well-suited for nonparametric methods which offer the important advantage that their results do not depend on model assumptions. For a general introduction to change point problems of this type, we refer to the monographs by Csörgő and Hórvath [13] and, with particular emphasis on nonparametric methods, to Brodsky and Darkhovsky [2].

Let $\mathbf{X}_1, \dots, \mathbf{X}_n$ denote a sample of a process $(\mathbf{X}_j)_{j \in \mathbb{Z}}$ with strictly stationary univariate margins, i.e., $X_{j,i} \sim F_i$ for all $j \in \mathbb{Z}$ and $i = 1, \dots, d$. We establish tests for the null hypothesis of a constant copula versus the alternative that there exist P unspecified change points $\lambda_1 < \dots < \lambda_P \in [0, 1]$, formally

$$\begin{aligned}
H_0 : & \mathbf{U}_j \sim C_1 \text{ for all } j = 1, \dots, n, \\
H_1 : & \text{there exist } 0 = \lambda_0 < \lambda_1 < \dots < \lambda_P < \lambda_{P+1} = 1 \text{ such that } \mathbf{U}_j \sim C_p \\
& \text{for all } j = \lfloor \lambda_{p-1}n \rfloor + 1, \dots, \lfloor \lambda_p n \rfloor \text{ and } p = 1, \dots, P+1,
\end{aligned}$$

where, under the alternative hypothesis, C_1, \dots, C_{P+1} are assumed to be pairwise different in at least one point $\mathbf{u} \in [0, 1]^d$. Unlike in the previous section, we estimate the pseudo-observations $\hat{\mathbf{U}}_1, \dots, \hat{\mathbf{U}}_n$ based on the whole sample $\mathbf{X}_1, \dots, \mathbf{X}_n$. The following test statistics are based on a comparison of the empirical distribution functions of the subsamples

$\widehat{\mathbf{U}}_1, \dots, \widehat{\mathbf{U}}_{\lfloor \zeta n \rfloor}$ and $\widehat{\mathbf{U}}_{\lfloor \zeta n \rfloor + 1}, \dots, \widehat{\mathbf{U}}_n$:

$$\mathbb{S}_n(\zeta, \mathbf{u}) := \frac{\lfloor \zeta n \rfloor (n - \lfloor \zeta n \rfloor)}{n^{3/2}} \left\{ \frac{1}{\lfloor \zeta n \rfloor} \sum_{j=1}^{\lfloor \zeta n \rfloor} \mathbf{1}_{\{\widehat{\mathbf{U}}_j \leq \mathbf{u}\}} - \frac{1}{n - \lfloor \zeta n \rfloor} \sum_{j=\lfloor \zeta n \rfloor + 1}^n \mathbf{1}_{\{\widehat{\mathbf{U}}_j \leq \mathbf{u}\}} \right\}.$$

Observe the similarity between S_n and the integrand in equation (16) of the previous section: except for the difference regarding the calculation of pseudo-observations, the functionals only differ by the factor $\sqrt{\lfloor \zeta n \rfloor (n - \lfloor \zeta n \rfloor)}/n$ which assigns less weight to change point candidates close to the sample's boundaries. Define $Z_n := \{1/n, \dots, (n-1)/n\}$. We consider three alternative test statistics which pick the most extreme realization within the set Z_n of change point candidates:

$$T_n^1 = \max_{\zeta \in Z_n} \int_{[0,1]^d} \mathbb{S}_n(\zeta, \mathbf{u})^2 d\widehat{C}_n(\mathbf{u}), \quad (19)$$

$$T_n^2 = \max_{\zeta \in Z_n} \left\{ \max_{\mathbf{u} \in \{\widehat{\mathbf{U}}_j\}_{j=1, \dots, n}} \mathbb{S}_n(\zeta, \mathbf{u}) - \min_{\mathbf{u} \in \{\widehat{\mathbf{U}}_j\}_{j=1, \dots, n}} \mathbb{S}_n(\zeta, \mathbf{u}) \right\}, \quad (20)$$

$$T_n^3 = \max_{\zeta \in Z_n} \left\{ \max_{\mathbf{u} \in \{\widehat{\mathbf{U}}_j\}_{j=1, \dots, n}} |\mathbb{S}_n(\zeta, \mathbf{u})| \right\}, \quad (21)$$

which are the *maximally selected Cramér-von Mises (CvM)*, *Kuiper (K)*, and *Kolmogorov-Smirnov (KS) statistic*, respectively. We refer to Hórvath and Shao [25] for an investigation of these statistics in a univariate context based on independent and identically distributed observations; T_n^3 is investigated in Inoue [26] for general multivariate distribution functions under strong mixing conditions as well as in Rémillard [38] with an application to the copula of GARCH residuals.

Theorem 5. *Consider a sample $\mathbf{X}_1, \dots, \mathbf{X}_n$ of a strictly stationary process $(\mathbf{X}_j)_{j \in \mathbb{Z}}$ satisfying the strong mixing condition $\alpha_{\mathbf{X}}(r) = \mathcal{O}(r^{-4-d\{1+\varepsilon\}})$ for some $0 < \varepsilon \leq 1/4$. Then under the null hypothesis, in $(\ell^\infty([0,1]^{d+1}), \|\cdot\|_\infty)$,*

$$\mathbb{S}_n(\zeta, \mathbf{u}) \xrightarrow{w.} \mathbb{S}_C(\zeta, \mathbf{u}) = \mathbb{B}_C(\zeta, \mathbf{u}) - \zeta \mathbb{B}_C(1, \mathbf{u}),$$

where $\mathbb{B}_C(\zeta, \mathbf{u})$ denotes a (centered) C -Kiefer process, with covariance structure

$$\text{Cov}(\mathbb{B}_C(\zeta_1, \mathbf{u}), \mathbb{B}_C(\zeta_2, \mathbf{v})) = \min(\zeta_1, \zeta_2) \sum_{j \in \mathbb{Z}} \text{Cov}(\mathbf{1}_{\{\mathbf{U}_0 \leq \mathbf{u}\}}, \mathbf{1}_{\{\mathbf{U}_j \leq \mathbf{v}\}})$$

for all $\zeta_1, \zeta_2 \in [0, 1]$ and $\mathbf{u}, \mathbf{v} \in [0, 1]^d$. This in particular implies weak convergence of the test statistics T_n^1 , T_n^2 , and T_n^3 under H_0 :

$$\begin{aligned} T_n^1 &\xrightarrow{w.} \sup_{0 \leq \zeta \leq 1} \int_{[0,1]^d} \{\mathbb{B}_C(\zeta, \mathbf{u}) - \zeta \mathbb{B}_C(1, \mathbf{u})\}^2 dC(\mathbf{u}), \\ T_n^2 &\xrightarrow{w.} \sup_{0 \leq \zeta \leq 1} \left[\sup_{\mathbf{u} \in [0,1]^d} \{\mathbb{B}_C(\zeta, \mathbf{u}) - \zeta \mathbb{B}_C(1, \mathbf{u})\} - \inf_{\mathbf{u} \in [0,1]^d} \{\mathbb{B}_C(\zeta, \mathbf{u}) - \zeta \mathbb{B}_C(1, \mathbf{u})\} \right], \\ T_n^3 &\xrightarrow{w.} \sup_{0 \leq \zeta \leq 1} \left[\sup_{\mathbf{u} \in [0,1]^d} |\{\mathbb{B}_C(\zeta, \mathbf{u}) - \zeta \mathbb{B}_C(1, \mathbf{u})\}| \right]. \end{aligned}$$

Similar calculations as in [25] reveal that $T_n^i \rightarrow \infty$ for $i = 1, 2, 3$ under H_1 . Hence, a test which rejects H_0 for unlikely large values of T_n^i is consistent against general alternatives. Approximate critical values of the tests can be derived from the tapered block multiplier technique.

Proposition 2. *Consider a sample $\mathbf{X}_1, \dots, \mathbf{X}_n$ of a process $(\mathbf{X}_j)_{j \in \mathbb{Z}}$ which satisfies $X_{j,i} \sim F_i$ for all $j \in \mathbb{Z}$ and $i = 1, \dots, d$. Further assume the process to fulfill the strong mixing assumptions of Theorem 3. Let $\xi_{1,n}, \dots, \xi_{n,n}$ denote a sample of a tapered block multiplier process $(\xi_{j,n})_{j \in \mathbb{Z}}$ satisfying **A1**, **A2**, **A3** with block length $l(n) \rightarrow \infty$, where $l(n) = \mathcal{O}(n^{1/2-\varepsilon})$ for $0 < \varepsilon < 1/2$ and define $\widehat{\mathbb{S}}_n(\zeta, \mathbf{u}) = \widehat{\mathbb{B}}_n^M(\zeta, \mathbf{u}) - \zeta \widehat{\mathbb{B}}_n^M(1, \mathbf{u})$, where*

$$\widehat{\mathbb{B}}_n^M(\zeta, \mathbf{u}) = n^{-1/2} \sum_{j=1}^{\lfloor \zeta n \rfloor} \left(\frac{\xi_{n,j}}{\xi_{\lfloor \zeta n \rfloor}} - 1 \right) \mathbf{1}_{\{\widehat{\mathbf{U}}_j \leq \mathbf{u}\}}.$$

Then, under the null hypothesis, $\widehat{\mathbb{S}}_n \xrightarrow[\xi]{\mathbb{P}} \mathbb{S}_C$, while under the alternative $\widehat{\mathbb{S}}_n = O_{\mathbb{P}}(1)$.

An application of the continuous mapping theorem proves consistency of the tapered block multiplier-based tests. The p-values of the test statistics are estimated as shown in Equation (18).

For simplicity, the change point location is assessed under the assumption that there is at most one change point. In this case, the alternative hypothesis can as well be formulated:

$$H_{1b} : \exists \lambda \in [0, 1] \text{ such that } \mathbf{U}_j \sim \begin{cases} C_1 & \text{for all } j = 1, \dots, \lfloor \lambda n \rfloor, \\ C_2 & \text{for all } j = \lfloor \lambda n \rfloor + 1, \dots, n, \end{cases}$$

where C_1 and C_2 are assumed to differ on a non-empty subset of $[0, 1]^d$. An estimator for the location of the change point $\widehat{\lambda}_n^i$, $i = 1, 2, 3$, is obtained by replacing max functions by arg max functions in Equations (19), (20), and (21). For ease of exposition, the superindex i is dropped in the following if no explicit reference to the functional is required. Given a (not necessarily correct) change-point estimator $\widehat{\lambda}_n$, the empirical copula of $\mathbf{X}_1, \dots, \mathbf{X}_{\lfloor \widehat{\lambda}_n n \rfloor}$ is an estimator of the unknown mixture distribution given by

$$C_{\widehat{\lambda}_n, 1}(\mathbf{u}) = \mathbf{1}_{\{\widehat{\lambda}_n \leq \lambda\}} C_1(\mathbf{u}) + \mathbf{1}_{\{\widehat{\lambda}_n > \lambda\}} \widehat{\lambda}_n^{-1} \left[\lambda C_1(\mathbf{u}) + (\widehat{\lambda}_n - \lambda) C_2(\mathbf{u}) \right] \quad (22)$$

[for an analogous estimator related to general distribution functions, see 10]. The latter coincides with C_1 if and only if the change point is estimated correctly. On the other hand, the empirical copula of $\mathbf{X}_{\lfloor \widehat{\lambda}_n n \rfloor + 1}, \dots, \mathbf{X}_n$ is an estimator of the unknown mixture distribution given by

$$C_{\widehat{\lambda}_n, 2}(\mathbf{u}) = \mathbf{1}_{\{\widehat{\lambda}_n \leq \lambda\}} (1 - \widehat{\lambda}_n)^{-1} \left[(\lambda - \widehat{\lambda}_n) C_1(\mathbf{u}) + (1 - \lambda) C_2(\mathbf{u}) \right] + \mathbf{1}_{\{\widehat{\lambda}_n > \lambda\}} C_2(\mathbf{u}), \quad (23)$$

for all $\mathbf{u} \in [0, 1]^d$. The latter coincides with C_2 if and only if the change point is estimated correctly. Consistency of $\widehat{\lambda}_n$ follows from consistency of the empirical copula and the fact that the difference of the two mixture distributions given in Equations (22) and (23) is

maximal in the case $\hat{\lambda}_n = \lambda$. Bai [1] iteratively applies the setting considered above to test for multiple breaks (one at a time), indicating a direction of future research to estimate locations of multiple change points in the dependence structure.

Finite sample properties. Size and power of the tests for a constant copula are shown in Tables 6 and 7 for $n = 400$ and $n = 800$, respectively. The observations are either serially independent or from a strictly stationary AR(1) process where the univariate innovations are either linked by a Clayton or a Gumbel–Hougaard copula. We consider the alternative hypothesis H_{1b} of at most one unspecified change point. If present, then this change point is located after observation $\lfloor \lambda n \rfloor = n/2$ and only affects the parameter within the investigated Clayton or Gumbel–Hougaard families: the copula C_1 is parameterized such that Kendall’s $\tau_1 = 0.2$, the copula C_2 such that Kendall’s $\tau_2 \in \{0.2, 0.6, 0.9\}$. We consider $S = 1,000$ ($n = 400$) or $S = 500$ ($n = 800$) tapered block multiplier simulations based on normal multiplier random variables with block length $l_M(400) = 5$, $l_M(800) = 6$ and kernel function κ_2 .

In the case of i.i.d. observations, we observe that the tapered block multiplier works similarly well as the standard multiplier (i.e., $l_M = 1$): the asymptotic size of the test, chosen to be 5%, is well approximated and its power increases in the difference $\tau_2 - \tau_1$. The estimated location of the change point, $\hat{\lambda}_n$, is close to its theoretical value. Moreover, its standard deviation $\hat{\sigma}(\hat{\lambda}_n)$ as well as its mean squared error $\text{MSE}(\hat{\lambda}_n)$ are decreasing in the difference $\tau_2 - \tau_1$. In the case of serially dependent observations sampled from AR(1) processes with $\beta = 0.25$, we find that the observed size of the test strongly deviates from its nominal size (chosen to be 5%) if serial dependence is neglected and the block length $l_M = 1$ is used: its estimates are reaching up to 18.7%. The test based on the tapered block multiplier with block length $l_M(400) = 5$ yields rejection quotas which approximate the asymptotic size well in all settings considered. These results are strengthened in Table 7 which shows results of MC simulations for sample size $n = 800$ and block length $l_M = 6$. The power improves considerably with the increased amount of observations and the change point location is well captured. Standard deviation and mean squared error of the estimated location of the change point, $\hat{\lambda}_n$, decrease in the difference $\tau_2 - \tau_1$.

Comparing the tests based on statistics T_n^1 , T_n^2 , and T_n^3 , we find that the test based on the Kuiper-type statistic performs best. The results indicate that the nominal size is well approximated in finite samples and that the test is most powerful in many settings. Likewise, with regard to the estimated location of the change point, the Kuiper-type statistic performs best in mean and in mean squared error.

The introduced tests for a constant copula offer some connecting factors for further research. For instance, Inoue [26] investigates nonparametric change point tests for the joint distribution of strongly mixing random vectors and finds that the observed size of the test heavily depends on the choice of the block length l in the resampling procedure. For different types of serially dependent observations, e.g., AR(1) processes with higher coefficient for the lagged variable or GARCH(1, 1) processes, it is of interest to investigate the optimal choice of the block length for the tapered block multiplier-based test with unspecified change point candidate. Moreover, test statistics based on different functionals offer potential for improvements. For instance, the Cramér-von Mises functional introduced by Rémillard and Scaillet [40] led to strong results in the case of a specified change point candidate. Though challenging from a computational point of view, an application of this functional to the case of unspecified change point candidate(s) is of interest as the functional yields very powerful tests.

Table 6: **Size and power of tests for a constant copula with unspecified change point candidate.** Results are based on 1,000 Monte Carlo replications, $n = 400$, $S = 1,000$, kernel function κ_2 , and $\alpha = 5\%$; additionally, the estimated change point location $\hat{\lambda}_n$, $\hat{\sigma}(\hat{\lambda}_n)$, and $MSE(\hat{\lambda}_n) \times 10^2$ are reported.

size/power						$\hat{\lambda}_n$		$\hat{\sigma}(\hat{\lambda}_n)$		$MSE(\hat{\lambda}_n)$		
			τ_2	0.2	0.6	0.9	0.6	0.9	0.6	0.9	0.6	0.9
i.i.d. setting												
Clayton	$l = 1$	CvM	0.061	0.406	0.873	0.511	0.506	0.093	0.061	0.871	0.372	
		K	0.040	0.495	0.991	0.496	0.496	0.074	0.048	0.556	0.228	
		KS	0.062	0.409	0.905	0.507	0.503	0.079	0.056	0.627	0.319	
	$l = 5$	CvM	0.035	0.342	0.847	0.519	0.509	0.084	0.058	0.735	0.349	
		K	0.031	0.375	0.983	0.495	0.496	0.070	0.049	0.489	0.246	
		KS	0.036	0.337	0.881	0.506	0.504	0.080	0.055	0.645	0.306	
	Gumbel	$l = 1$	CvM	0.047	0.456	0.903	0.509	0.507	0.083	0.055	0.694	0.304
			K	0.056	0.474	0.992	0.494	0.495	0.074	0.050	0.548	0.251
			KS	0.052	0.437	0.910	0.502	0.504	0.080	0.054	0.644	0.291
$l = 5$		CvM	0.043	0.387	0.880	0.506	0.507	0.086	0.055	0.739	0.310	
		K	0.026	0.343	0.974	0.487	0.496	0.072	0.046	0.518	0.217	
		KS	0.041	0.349	0.884	0.497	0.503	0.079	0.056	0.642	0.312	
AR(1) setting with $\beta = 0.25$												
Clayton	$l = 1$	CvM	0.187	0.487	0.846	0.519	0.510	0.116	0.075	1.390	0.566	
		K	0.139	0.562	0.994	0.490	0.494	0.084	0.052	0.707	0.276	
		KS	0.177	0.499	0.913	0.504	0.506	0.098	0.071	0.969	0.503	
	$l = 5$	CvM	0.046	0.243	0.697	0.516	0.512	0.102	0.070	1.073	0.497	
		K	0.039	0.289	0.954	0.496	0.493	0.073	0.056	0.535	0.315	
		KS	0.040	0.253	0.771	0.506	0.506	0.095	0.065	0.901	0.421	
	Gumbel	$l = 1$	CvM	0.185	0.536	0.878	0.515	0.513	0.119	0.079	1.443	0.649
			K	0.123	0.576	0.997	0.487	0.493	0.086	0.051	0.746	0.272
			KS	0.167	0.541	0.913	0.504	0.509	0.103	0.075	1.078	0.573
$l = 5$		CvM	0.042	0.295	0.706	0.519	0.506	0.096	0.074	0.949	0.547	
		K	0.040	0.294	0.939	0.495	0.495	0.084	0.053	0.716	0.282	
		KS	0.050	0.287	0.745	0.509	0.504	0.089	0.067	0.793	0.457	

Table 7: **Size and power of tests for a constant copula with unspecified change point candidate.** Results are based on 1,000 Monte Carlo replications, $n = 800$, $S = 500$, kernel function κ_2 , and $\alpha = 5\%$; additionally, the estimated change point location $\hat{\lambda}_n$, $\hat{\sigma}(\hat{\lambda}_n)$, and $MSE(\hat{\lambda}_n) \times 10^2$ are reported.

size/power						$\hat{\lambda}_n$		$\hat{\sigma}\left(\hat{\lambda}_n\right)$		$MSE\left(\hat{\lambda}_n\right)$		
		τ_2	0.2	0.6	0.9	0.6	0.9	0.6	0.9	0.6	0.9	
i.i.d. setting												
Clayton	$l=1$	CvM	0.033	0.695	0.999	0.508	0.506	0.079	0.041	0.623	0.173	
		K	0.054	0.901	1.000	0.494	0.497	0.059	0.029	0.350	0.087	
		KS	0.046	0.734	0.999	0.505	0.503	0.068	0.038	0.463	0.142	
	$l=6$	CvM	0.043	0.681	0.997	0.510	0.505	0.076	0.042	0.585	0.177	
		K	0.037	0.838	1.000	0.494	0.497	0.057	0.031	0.335	0.096	
		KS	0.036	0.699	0.998	0.507	0.503	0.062	0.039	0.384	0.149	
	Gumbel	$l=1$	CvM	0.049	0.510	0.953	0.510	0.506	0.090	0.049	0.823	0.243
			K	0.037	0.688	1.000	0.490	0.496	0.061	0.032	0.372	0.106
			KS	0.051	0.509	0.966	0.502	0.503	0.081	0.044	0.672	0.199
$l=6$		CvM	0.055	0.721	0.998	0.505	0.504	0.069	0.037	0.477	0.138	
		K	0.034	0.775	1.000	0.498	0.496	0.059	0.029	0.355	0.086	
		KS	0.045	0.686	0.998	0.504	0.500	0.065	0.039	0.428	0.156	
AR(1) setting with $\beta=0.25$												
Clayton	$l=1$	CvM	0.184	0.712	0.995	0.519	0.507	0.097	0.056	0.973	0.321	
		K	0.122	0.914	1.000	0.495	0.497	0.063	0.034	0.411	0.122	
		KS	0.169	0.756	0.999	0.512	0.506	0.084	0.050	0.714	0.248	
	$l=6$	CvM	0.065	0.488	0.939	0.506	0.508	0.089	0.054	0.803	0.293	
		K	0.057	0.724	1.000	0.493	0.496	0.060	0.033	0.359	0.113	
		KS	0.062	0.521	0.965	0.500	0.503	0.072	0.045	0.527	0.206	
	Gumbel	$l=1$	CvM	0.207	0.760	0.992	0.507	0.509	0.091	0.052	0.841	0.273
			K	0.141	0.879	1.000	0.489	0.497	0.070	0.036	0.496	0.135
			KS	0.182	0.780	0.998	0.500	0.505	0.089	0.047	0.805	0.218
$l=6$		CvM	0.052	0.533	0.959	0.514	0.508	0.083	0.052	0.707	0.273	
		K	0.046	0.670	1.000	0.495	0.496	0.065	0.032	0.426	0.106	
		KS	0.053	0.520	0.974	0.508	0.505	0.076	0.048	0.584	0.232	

4. Conclusion

Consistent tests for constancy of the copula with specified or unspecified change point candidate are introduced. We observe a trade-off in assumptions required for the testing: if a change point candidate is specified, then the test is consistent whether or not there is a simultaneous change point in marginal distribution function(s). If change point candidate(s) are unspecified, then the assumption of strictly stationary marginal distribution functions is required and allows to drop continuity assumptions on the partial derivatives of the underlying copula(s). Tests are shown to behave well in size and power when applied to various types of dependent observations. P-Values of the tests are estimated using a tapered block multiplier technique which is based on serially dependent multiplier random variables; the latter is shown to perform better than the block bootstrap in mean and mean squared error when estimating the asymptotic covariance structure of the empirical copula process in various settings.

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Appendix A. Proofs

Proof of Theorem 1. Since $\hat{F}_i(X_{j,i}) \leq u_i$ if and only if $\hat{G}_i(U_{j,i}) \leq u_i$, where \hat{G}_i denotes the empirical distribution function of $U_{1,i}, \dots, U_{n,i}$, we can assume without loss of generality, that $\mathbf{X}_j = \mathbf{U}_j \sim C$. Weak convergence of $\mathbb{B}_{C,n}$ under the strong mixing condition $\alpha_{\mathbf{X}}(r) = \mathcal{O}(r^{-a})$ for some $a > 1$ is established in Rio [41]. Thus, Condition 2.1 in Bücher and Volgushev [6] is satisfied and an application of the functional delta method and of Theorem 2.4 in [6] yields

$$\sqrt{n}\{C_n(C_{n,1}^{-1}, \dots, C_{n,d}^{-1}) - C\} \xrightarrow{w.} \mathbb{G}_C,$$

where $C_{n,i}$ denotes the i -th marginal of C_n , $i = 1, \dots, d$. The assertion follows from $\|\hat{\mathbb{G}}_{C,n} - \sqrt{n}\{C_n(C_{n,1}^{-1}, \dots, C_{n,d}^{-1}) - C\}\|_{\infty} = O(n^{-1/2})$. ■

Proof of Theorem 2. Without loss of generality we may assume $\mathbf{X}_j = \mathbf{U}_j$. Let $C_{n,b}$ denote the empirical distribution function of the sample

$$\mathbf{U}_{H_1+1}, \dots, \mathbf{U}_{H_1+l_B}, \mathbf{U}_{H_2+1}, \dots, \mathbf{U}_{H_2+l_B}, \dots, \mathbf{U}_{H_k+1}, \dots, \mathbf{U}_{H_k+l_B}.$$

Again, $\|\hat{\mathbb{G}}_{C,n}^B - \sqrt{n}\{C_{n,b}(C_{n,b,1}^{-1}, \dots, C_{n,b,d}^{-1}) - C_n\}\|_{\infty} = O(n^{-1/2})$, and the result follows from Corollary 2.11 in [6]. ■

Proof of Theorem 3. Without loss of generality we may assume $\mathbf{X}_j = \mathbf{U}_j$. The process

$$\mathbb{B}_{C,n}^M(\mathbf{u}) = \sqrt{n} \left\{ \frac{1}{n} \sum_{j=1}^n \frac{\xi_{n,j}}{\bar{\xi}_n} \mathbf{1}_{\{\mathbf{U}_j \leq \mathbf{u}\}} - C_n(\mathbf{u}) \right\} = \frac{1}{\sqrt{n}} \sum_{j=1}^n \left(\frac{\xi_{n,j}}{\bar{\xi}_n} - 1 \right) \left\{ \mathbf{1}_{\{\mathbf{U}_j \leq \mathbf{u}\}} - C(\mathbf{u}) \right\}$$

is defined on a product space $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mathbb{P}_1 \otimes \mathbb{P}_2)$, where \mathbf{X}_j is defined on $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ and $\xi_{n,j}$ on $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$ for all $j = 1, \dots, n$. It follows from the results in Bühlmann [7, Section 3.3] and [50, Section 1.5] that there exists a set $A_1 \in \mathcal{F}_1$ with $\mathbb{P}_1(A_1) = 1$ such that

$$\mathbb{B}_{C,n}^M(\omega_1, \cdot) \xrightarrow{w} \mathbb{B}_C \text{ in } \ell^\infty([0, 1]^d) \text{ for all } \omega_1 \in A_1,$$

where \xrightarrow{w} denotes weak convergence with respect to the multipliers. By Theorem 1.5.7 and its addendum in [50], $\mathbb{B}_{C,n}^M(\omega_1, \cdot)$ is asymptotically uniformly equicontinuous for each such ω_1 . As $\sup_{u_i \in [0, 1]} |\hat{F}_i^-(u_i) - u_i| = o(1)$, \mathbb{P}_1 -almost surely, we obtain

$$\tilde{\mathbb{B}}_{C,n}^M(\mathbf{u}) = \mathbb{B}_{C,n}^M(\hat{F}_1^{-1}(u_1), \dots, \hat{F}_d^{-1}(u_d)) \xrightarrow{w} \mathbb{B}_C \quad (\mathbb{P}_1\text{-a.s.}).$$

Note that

$$\|\tilde{\mathbb{B}}_{C,n}^M - \hat{\mathbb{B}}_{C,n}^M\|_\infty = O(n^{-1/2} \max_{j=1}^n |\xi_{n,j}/\bar{\xi}_n|) = o_{\mathbb{P}_2}(1),$$

where the last estimation follows from $\mathbb{P}_2(|\xi_{n,1}| > x) = o(x^{-2})$ and $\mathbb{P}_2(\max_{j=1}^n |\xi_{n,j}| > \varepsilon \sqrt{n} \bar{\xi}_n) \leq \mathbb{P}_2(\bar{\xi}_n \leq 1/2) + n\mathbb{P}_2(|\xi_{n,1}| > \varepsilon \sqrt{n}/2)$. Thus $\hat{\mathbb{B}}_{C,n}^M \xrightarrow{w} \mathbb{B}_C$, \mathbb{P}_1 -almost surely, and since $\hat{\mathbb{B}}_{C,n}^M$ is a ball-measurable element of the space of cadlag functions $(D([0, 1]^d), \|\cdot\|_\infty) \subset (\ell^\infty([0, 1]^d), \|\cdot\|_\infty)$, a detailed look at Theorem 1.7.2 and the proof of Theorem 2.9.6 in [50] allows to translate this result into the Hoffmann-Jørgensen weak convergence, i.e.,

$$\hat{\mathbb{B}}_{C,n}^M \xrightarrow[\xi]{\mathbb{P}} \mathbb{B}_C \text{ in } (\ell^\infty([0, 1]^d), \|\cdot\|_\infty).$$

Finally, set

$$\tilde{\mathbb{G}}_{C,n}^M(\mathbf{u}) = \hat{\mathbb{B}}_{C,n}^M(\mathbf{u}) - \sum_{i=1}^d D_i C(\mathbf{u}) \hat{\mathbb{B}}_{C,n}^M(\mathbf{u}^{(i)}).$$

Proceeding as in the proof of Proposition 3.2 in [47] we obtain $\|\tilde{\mathbb{G}}_{C,n}^M - \hat{\mathbb{G}}_{C,n}^M\|_\infty = o_P(1)$, which yields the assertion of the Theorem after an application of Lemma B.1 in [5]. ■

Proof of Theorem 4. Without loss of generality we may assume $\mathbf{X}_j = \mathbf{U}_j$. We begin by proving the joint weak convergence

$$(\mathbb{B}_{C,n,1}, \mathbb{B}_{C,n,2}) \xrightarrow{w} (\mathbb{B}_{C,1}, \mathbb{B}_{C,2}) \tag{A.1}$$

in $\ell^\infty([0, 1]^d) \times \ell^\infty([0, 1]^d)$, where

$$\begin{aligned} \mathbb{B}_{C,n,1}(\mathbf{u}) &= \frac{1}{\sqrt{[\lambda n]}} \sum_{j=1}^{[\lambda n]} \left\{ \mathbf{1}_{\{\mathbf{U}_j \leq \mathbf{u}\}} - C(\mathbf{u}) \right\} \\ \mathbb{B}_{C,n,2}(\mathbf{u}) &= \frac{1}{\sqrt{n - [\lambda n]}} \sum_{j=[\lambda n]+1}^n \left\{ \mathbf{1}_{\{\mathbf{U}_j \leq \mathbf{u}\}} - C(\mathbf{u}) \right\}. \end{aligned}$$

Asymptotic tightness of $(\mathbb{B}_{C,n,1}, \mathbb{B}_{C,n,2})$ follows from Lemma 1.4.3 in [50] and the fact that

$$\mathbb{B}_{C,n,2} = \sqrt{\frac{1}{1-s}} \mathbb{B}_{C,n} - \sqrt{\frac{s}{1-s}} \mathbb{B}_{C,n,1} + o_{\mathbb{P}}(1),$$

where both processes on the right-hand side are asymptotically tight by Theorem 1. It therefore suffices to show that

$$(\mathbb{B}_{C,n,1}(\mathbf{u}_1), \dots, \mathbb{B}_{C,n,1}(\mathbf{u}_k), \mathbb{B}_{C,n,2}(\mathbf{v}_1), \dots, \mathbb{B}_{C,n,2}(\mathbf{v}_l)) \\ \xrightarrow{w.} (\mathbb{B}_{C,1}(\mathbf{u}_1), \dots, \mathbb{B}_{C,1}(\mathbf{u}_k), \mathbb{B}_{C,2}(\mathbf{v}_1), \dots, \mathbb{B}_{C,2}(\mathbf{v}_l))$$

in \mathbb{R}^{k+l} , see Problem 1.5.3 in [50]. For the sake of a clear exposition we only consider the case $k = l = 1$, the general case follows along similar lines. We have to prove that, for each $c = (c_1, c_2) \in \mathbb{R}^2$,

$$c_1 \mathbb{B}_{C,n,1}(\mathbf{u}) + c_2 \mathbb{B}_{C,n,2}(\mathbf{v}) \xrightarrow{w.} c_1 \mathbb{B}_{C,1}(\mathbf{u}) + c_2 \mathbb{B}_{C,2}(\mathbf{v}).$$

Setting $Z_j(\mathbf{u}) = \mathbf{1}_{\{\mathbf{U}_j \leq \mathbf{u}\}} - C(\mathbf{u})$, the left-hand side of the previous expression can be written as $\sum_{j=1}^n \zeta_{n,j}$, where

$$\zeta_{n,j} = \frac{c_1}{\sqrt{[\lambda n]}} Z_j(\mathbf{u}) \mathbf{1}_{\{j \leq [\lambda n]\}} + \frac{c_2}{\sqrt{n - [\lambda n]}} Z_j(\mathbf{v}) \mathbf{1}_{\{j > [\lambda n]\}}.$$

The asserted weak convergence follows from Theorem 2.1 in [34], if we prove the corresponding conditions of that Theorem. We have

$$\mathbb{E} \left[\left(\sum_{j=1}^n \zeta_{n,j} \right)^2 \right] = \frac{c_1^2}{[\lambda n]} \sum_{i,j=1}^{[\lambda n]} \mathbb{E}[Z_i(\mathbf{u}) Z_j(\mathbf{u})] + \frac{c_2^2}{n - [\lambda n]} \sum_{i,j=[\lambda n]+1}^n \mathbb{E}[Z_i(\mathbf{v}) Z_j(\mathbf{v})] \\ + \frac{2c_1 c_2}{\sqrt{[\lambda n](n - [\lambda n])}} \sum_{i=1}^{[\lambda n]} \sum_{j=[\lambda n]+1}^n \mathbb{E}[Z_i(\mathbf{u}) Z_j(\mathbf{v})].$$

The first two sums converge to $c_1^2 \gamma(\mathbf{u}, \mathbf{u})$ and $c_2^2 \gamma(\mathbf{v}, \mathbf{v})$, respectively. Suppose that $[\lambda n] > n - [\lambda n]$, the opposite case is treated analogously. Then, a tedious calculation shows that the third sum in the previous identity equals

$$\frac{2c_1 c_2}{\sqrt{[\lambda n](n - [\lambda n])}} \left\{ \sum_{k=1}^{n-[\lambda n]} k r(k) + \sum_{k=n-[\lambda n]+1}^{[\lambda n]-1} (n - [\lambda n]) r(k) + \sum_{k=[\lambda n]}^{n-1} (n - k) r(k) \right\},$$

where $r(|i - j|) = \mathbb{E}[Z_i(\mathbf{u}) Z_j(\mathbf{v})]$. The first sum in the curly bracket multiplied with n^{-1} converges to 0 by dominated convergence. If we multiply the other two sums with n^{-1} , then we obtain tails of absolutely converging series, which also converge to 0. To conclude,

$$\mathbb{E} \left[\left(\sum_{j=1}^n \zeta_{n,j} \right)^2 \right] \rightarrow c^T \begin{pmatrix} \gamma(\mathbf{u}, \mathbf{u}) & 0 \\ 0 & \gamma(\mathbf{v}, \mathbf{v}) \end{pmatrix} c = \text{Var}(c_1 \mathbb{B}_{C,1}(\mathbf{u}) + c_2 \mathbb{B}_{C,2}(\mathbf{v})).$$

Since $|\zeta_{n,j}| = O(n^{-1/2})$ uniformly in j , some easy calculations show that condition (2.1) and (2.2) in [34] are satisfied. Hence, by Theorem 2.1 in that reference, the asserted weak convergence follows.

Define

$$\begin{aligned}\widehat{\mathbb{G}}_{C,n,1}(\mathbf{u}) &= \frac{1}{\sqrt{[\lambda n]}} \sum_{j=1}^{[\lambda n]} \left\{ \mathbf{1}_{\{\widehat{\mathbf{U}}_j \leq \mathbf{u}\}} - C(\mathbf{u}) \right\} \\ \widehat{\mathbb{G}}_{C,n,2}(\mathbf{u}) &= \frac{1}{\sqrt{n - [\lambda n]}} \sum_{j=[\lambda n]+1}^n \left\{ \mathbf{1}_{\{\widehat{\mathbf{V}}_j \leq \mathbf{u}\}} - C(\mathbf{u}) \right\}.\end{aligned}$$

An application of the functional delta method to $\mathbb{B}_{C,n,p}$ with the mapping Φ from Theorem 2.4 in [6] (and the usual estimation of the remainder term descending from the fact that $F_{nj}(X_{ij}) \leq u_j$ and $X_{ij} \leq F_{nj}^{-1}(u_j)$ are not equivalent) yields

$$(\widehat{\mathbb{G}}_{C,n,1}, \widehat{\mathbb{G}}_{C,n,2}) \xrightarrow{w.} (\mathbb{G}_{C,1}, \mathbb{G}_{C,2}) \quad \text{in } \ell^\infty([0, 1]^d) \times \ell^\infty([0, 1]^d).$$

Hence, Theorem 4 follows from the continuous mapping theorem observing that, under H_0 ,

$$\sqrt{\frac{[\lambda n](n - [\lambda n])}{n}} \left\{ \widehat{C}_{1, \dots, [\lambda n]} - \widehat{C}_{n - [\lambda n], \dots, n} \right\} = \sqrt{\frac{n - [\lambda n]}{n}} \widehat{\mathbb{G}}_{C,n,1} - \sqrt{\frac{[\lambda n]}{n}} \widehat{\mathbb{G}}_{C,n,2}.$$

■

Proof of Proposition 1. Without loss of generality we may assume $\mathbf{X}_j = \mathbf{U}_j$. We consider the processes

$$\begin{aligned}\mathbb{B}_{C,n,1}^M(\mathbf{u}) &= \frac{1}{\sqrt{[\lambda n]}} \sum_{j=1}^{[\lambda n]} \left(\frac{\xi_{n,j}}{\bar{\xi}_n^{(1)}} - 1 \right) \mathbf{1}_{\{\mathbf{U}_j \leq \mathbf{u}\}} \\ &= \frac{1}{\sqrt{[\lambda n]}} \sum_{j=1}^{[\lambda n]} \left(\frac{\xi_{n,j}}{\bar{\xi}_n^{(1)}} - 1 \right) \left\{ \mathbf{1}_{\{\mathbf{U}_j \leq \mathbf{u}\}} - C(\mathbf{u}) \right\} \\ \mathbb{B}_{C,n,2}^M(\mathbf{u}) &= \frac{1}{\sqrt{n - [\lambda n]}} \sum_{j=[\lambda n]+1}^n \left(\frac{\xi_{n,j}}{\bar{\xi}_n^{(2)}} - 1 \right) \mathbf{1}_{\{\mathbf{U}_j \leq \mathbf{u}\}} \\ &= \frac{1}{\sqrt{n - [\lambda n]}} \sum_{j=[\lambda n]+1}^n \left(\frac{\xi_{n,j}}{\bar{\xi}_n^{(2)}} - 1 \right) \left\{ \mathbf{1}_{\{\mathbf{U}_j \leq \mathbf{u}\}} - C(\mathbf{u}) \right\}\end{aligned}$$

which are defined on the product space $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mathbb{P}_1 \otimes \mathbb{P}_2)$, where \mathbf{X}_j is defined on $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ and $\xi_{n,j}$ on $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$ for all $j = 1, \dots, n$. For the subsequent investigations, we may replace the arithmetic means $\bar{\xi}_n^{(p)}$ by $1 = \mathbb{E}\xi_{n,j}$. As in the proof of Theorem 3, the results in Bühlmann [7, Section 3.3] guarantee the existence of a set $A_1 \in \mathcal{F}_1$ with $\mathbb{P}_1(A_1) = 1$ such that both

$$\mathbb{B}_{C,n,1}^M(\omega_1, \cdot) \xrightarrow{w.} \mathbb{B}_{C,1}, \quad \mathbb{B}_{C,n,2}^M(\omega_1, \cdot) \xrightarrow{w.} \mathbb{B}_{C,2} \quad \text{for all } \omega_1 \in A_1.$$

(use an index shift for $\mathbb{B}_{C,n,2}$), where $\xrightarrow{w.}$ denotes weak convergence with respect to the multipliers. In the following we show joint weak convergence with respects to the multipliers, \mathbb{P}_1 -almost surely. Joint asymptotic tightness follows from asymptotic tightness of the components. As in the proof of Theorem 4 it remains to show

$$\begin{aligned}(\mathbb{B}_{C,n,1}^M(\mathbf{u}_1), \dots, \mathbb{B}_{C,n,1}^M(\mathbf{u}_k), \mathbb{B}_{C,n,2}^M(\mathbf{v}_1), \dots, \mathbb{B}_{C,n,2}^M(\mathbf{v}_l))(\omega_1, \cdot) \\ \xrightarrow{w.} (\mathbb{B}_{C,1}(\mathbf{u}_1), \dots, \mathbb{B}_{C,1}(\mathbf{u}_k), \mathbb{B}_{C,2}(\mathbf{v}_1), \dots, \mathbb{B}_{C,2}(\mathbf{v}_l))\end{aligned}$$

in \mathbb{R}^{k+l} , which can be done by Theorem 2.1 in [34]. Again, for the sake of a clear exposition, we only consider the case $k = l = 1$. We have to prove that, for each $c = (c_1, c_2) \in \mathbb{R}^2$,

$$c_1 \mathbb{B}_{C,n,1}^M(\mathbf{u}) + c_2 \mathbb{B}_{C,n,2}^M(\mathbf{v}) \xrightarrow{w.} c_1 \mathbb{B}_{C,1}(\mathbf{u}) + c_2 \mathbb{B}_{C,2}(\mathbf{v}) \quad (\mathbb{P}_1\text{-a.s.}).$$

The left-hand side of this display can be written as $\sum_{j=1}^n \zeta_{n,j}$, where

$$\zeta_{n,j} = \frac{c_1}{\sqrt{[\lambda n]}} \mathbf{1}_{\{\mathbf{U}_j \leq \mathbf{u}\}} \mathbf{1}_{\{j \leq [\lambda n]\}} + \frac{c_2}{\sqrt{n - [\lambda n]}} \mathbf{1}_{\{\mathbf{U}_j \leq \mathbf{v}\}} \mathbf{1}_{\{j > [\lambda n]\}}.$$

An easy calculation shows that, uniformly in \mathbf{u}, \mathbf{v} and ω_1 ,

$$\begin{aligned} |\mathbb{E}_2 [\mathbb{B}_{C,n,1}^M(\mathbf{u}) \mathbb{B}_{C,n,2}^M(\mathbf{v})]| &\leq \frac{1}{\sqrt{[\lambda n](n - [\lambda n])}} \sum_{i=1}^{[\lambda n]} \sum_{j=[\lambda n]+1}^n |\text{Cov}(\xi_{n,i}, \xi_{n,j})| \\ &\leq \text{const} \frac{1}{n} \sum_{i=1}^{c \cdot l(n)-1} i \left| v \left(\frac{i}{l(n)} \right) \right| = o(1). \end{aligned}$$

Therefore, after some similar calculations as in the proof of Theorem 4, $\sigma_n^2 = \mathbb{E}_2[(\sum_{j=1}^n \zeta_{n,j})^2] \rightarrow \text{Var}(c_1 \mathbb{B}_{C,1}(\mathbf{u}) + c_2 \mathbb{B}_{C,2}(\mathbf{v}))$, \mathbb{P}_1 -almost surely. Since $|\zeta_{n,j}| = O(|\xi_{n,j} - 1| \cdot n^{-1/2})$ uniformly in \mathbf{u}, \mathbf{v} and ω_1 , condition (2.1) in [34] is satisfied \mathbb{P}_1 -almost surely. Finally, we check the Lindeberg condition (2.2) in [34], i.e., for all $\varepsilon > 0$,

$$\frac{1}{\sigma_n^2} \sum_{j=1}^n \mathbb{E}_2[\zeta_{n,j}^2 \mathbf{1}_{\{|\zeta_{n,j}|/\sigma_n > \varepsilon\}}] \rightarrow 0 \quad (\mathbb{P}_1\text{-a.s.})$$

for $n \rightarrow \infty$. The expression on the left-hand side of this display can be estimated uniformly in \mathbf{u}, \mathbf{v} and ω_1 by

$$\text{const} \cdot \mathbb{E}_2[(\xi_{n,1} - 1)^2 \mathbf{1}_{\{|\xi_{n,1} - 1| > \text{const} \sqrt{n}\}}] \leq \text{const} \cdot \frac{1}{n} \mathbb{E}_2[(\xi_{n,1} - 1)^4],$$

which yields the assertion by the assumption on the finite centered moments of $\xi_{n,j}$. To conclude,

$$(\mathbb{B}_{C,n,1}^M, \mathbb{B}_{C,n,2}^M) \xrightarrow{w.} (\mathbb{B}_{C,1}, \mathbb{B}_{C,2}) \quad (\mathbb{P}_1\text{-a.s.})$$

and analogously to the proof of Theorem 3 this implies

$$(\widehat{\mathbb{B}}_{C,n,1}^M, \widehat{\mathbb{B}}_{C,n,2}^M) \xrightarrow{w.} (\mathbb{B}_{C,1}, \mathbb{B}_{C,2}) \quad (\mathbb{P}_1\text{-a.s.}).$$

Thus, by the continuous mapping Theorem

$$\begin{aligned} \sqrt{1-\lambda} \left\{ \widehat{\mathbb{B}}_{C,n,1}^M - \sum_{i=1}^d D_i C(\mathbf{u}) \widehat{\mathbb{B}}_{C,n,1}^M(\mathbf{u}^{(i)}) \right\} - \sqrt{\lambda} \left\{ \widehat{\mathbb{B}}_{C,n,1}^M - \sum_{i=1}^d D_i C(\mathbf{u}) \widehat{\mathbb{B}}_{C,n,1}^M(\mathbf{u}^{(i)}) \right\} \\ \xrightarrow{w.} \sqrt{1-\lambda} \mathbb{G}_{C,1}(\mathbf{u}) - \sqrt{\lambda} \mathbb{G}_{C,2}(\mathbf{u}) \quad (\mathbb{P}_1\text{-a.s.}). \end{aligned}$$

Again, the expression on the left-hand side is a ball-measurable element of $D([0, 1]^d)$ which allows to translate the latter weak convergence into the Hoffmann-Jørgensen conditional weak convergence. Finally, replace the true partial derivatives by their estimators and note

that the difference is uniformly $o_P(1)$, which yields the assertion after an application of Lemma B.1 in [5]. ■

Proof of Theorem 5. Without loss of generality we may assume $\mathbf{X}_j = \mathbf{U}_j$. Under the conditions on the mixing rate, it follows from Theorem 2 in Philipp and Pinzur [36] that the process

$$\mathbb{B}_{C,n}(\zeta, \mathbf{u}) := \frac{1}{\sqrt{n}} \sum_{j=1}^{\lfloor \zeta n \rfloor} \left\{ \mathbf{1}_{\{\mathbf{X}_j \leq \mathbf{u}\}} - C(\mathbf{u}) \right\}$$

weakly converges in $\ell^\infty([0, 1]^{d+1})$ to a C -Kiefer process $\mathbb{B}_C(\zeta, \mathbf{u})$. Hence, Condition 3.1 in [6] is satisfied and an application of Corollary 3.3 in that reference yields that

$$\tilde{\mathbb{S}}_n(\zeta, \mathbf{u}) = \frac{1}{\sqrt{n}} \sum_{j=1}^{\lfloor \zeta n \rfloor} \left\{ \mathbf{1}_{\{X_{j1} \leq F_{n1}^{-1}(u_1), \dots, X_{jd} \leq F_{nd}^{-1}(u_d)\}} - C_n(C_{n1}^{-1}(u_1), \dots, C_{nd}^{-1}(u_d)) \right\}$$

weakly converges to $\mathbb{S}_C(\zeta, \mathbf{u}) = \mathbb{B}_C(\zeta, \mathbf{u}) - \zeta \mathbb{B}_C(1, \mathbf{u})$ in $\ell^\infty([0, 1]^{d+1})$. Under the null hypothesis we can write

$$\mathbb{S}_n(\zeta, \mathbf{u}) = \frac{1}{\sqrt{n}} \sum_{j=1}^{\lfloor \zeta n \rfloor} \left\{ \mathbf{1}_{\{\hat{\mathbf{U}}_j \leq \mathbf{u}\}} - \hat{C}_n(\mathbf{u}) \right\}$$

and the assertion follows from $\|\tilde{\mathbb{S}}_n - \mathbb{S}_n\|_\infty = O_P(n^{-1/2})$. ■

Proof of Proposition 2. Without loss of generality we may assume $\mathbf{X}_j = \mathbf{U}_j$. Suppose the null hypothesis holds and define

$$\mathbb{B}_{C,n}^M(\zeta, \mathbf{u}) = \frac{1}{\sqrt{n}} \sum_{j=1}^{\lfloor \zeta n \rfloor} \left(\frac{\xi_{n,j}}{\bar{\xi}_{\lfloor \zeta n \rfloor}} - 1 \right) \mathbf{1}_{\{\mathbf{X}_j \leq \mathbf{u}\}} = \frac{1}{\sqrt{n}} \sum_{j=1}^{\lfloor \zeta n \rfloor} \left(\frac{\xi_{n,j}}{\bar{\xi}_{\lfloor \zeta n \rfloor}} - 1 \right) \left\{ \mathbf{1}_{\{\mathbf{X}_j \leq \mathbf{u}\}} - C(\mathbf{u}) \right\}.$$

We may replace $\bar{\xi}_{\lfloor \zeta n \rfloor}$ by 1. Similar as in the proofs of Proposition 1 and Theorem 3 we begin by showing that $\mathbb{B}_{C,n}^M \xrightarrow{w.} \mathbb{B}_C$, \mathbb{P}_1 -almost surely, where $\xrightarrow{w.}$ denotes weak convergence with respect to the multipliers. Convergence of the finite dimensional distributions follows along similar lines as in the previous proofs; the details are omitted for the sake of brevity. Tightness follows by tightness of $\mathbb{B}_{C,n}^M(1, \cdot)$ and by analogous arguments as in the proof of Theorem 2.12.1 in [50].

Thus, as in the proof of Theorem 3, $\widehat{\mathbb{B}}_{C,n}^M \xrightarrow[\xi]{\mathbb{P}} \mathbb{B}_C$; and hence $\widehat{\mathbb{S}}_n^M \xrightarrow[\xi]{\mathbb{P}} \mathbb{S}$ by the Lipschitz-continuous mapping Theorem for the bootstrap, see Proposition 10.7 in [29]. Under the alternative, again replacing $\bar{\xi}_{\lfloor \zeta n \rfloor}$ by 1, we have

$$\frac{1}{n} \mathbb{E}_2 \left(\sum_{j=1}^{\lfloor \zeta n \rfloor} (\xi_{n,j} - 1) \mathbf{1}_{\{\mathbf{X}_j \leq \mathbf{u}\}} \right)^2 \leq \frac{l_n}{n} \sum_{k=1}^{l_n} (1 - k/l_n) v(k/l_n) = o(1)$$

by dominated convergence, which yields $\|\widehat{\mathbb{B}}_{C,n}^M(\zeta, \mathbf{u})\|_\infty = o_{\mathbb{P}_2}(1)$ uniformly in ζ, \mathbf{u} and ω_1 . ■

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